

Industrial Asset Pricing*

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Abstract

We develop a dynamic general equilibrium model with many different industries, in which firms set prices strategically in product markets. Our approach extends previous literature by endogenizing the market price of risk and allowing for more general risk structures. General equilibrium in the model is shown to exist under general conditions. Strategic interaction between firms amplifies business cycles, small changes in long-term growth rates can have drastic effects on the equilibrium outcome, whereas temporary changes in productivity have marginal impact, and the overall competitiveness of the economy only depends on long-term growth. A firm's expected returns are affected by the industrial environment in which it operates, in line with what has been observed in the empirical literature. Overall, our model suggests that industry characteristics should be informative about the expected returns of individual firms over the business cycle.

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1 Introduction

In 1973 the OPEC cartel raised oil prices, precipitating a global recession. This event was unique but not unprecedented: Several of the largest shocks in financial markets as well as in the real economy in the last century originated from the strategic behavior of firms and other entities. The general equilibrium asset pricing literature does not speak to such strategic interactions. Typically, in such models, shocks to the economy are “technological,” driven by exogenous processes. However, history suggests that technological shocks, although important, do not capture the whole story. Understanding the general equilibrium effects of such strategic behavior should therefore be an important goal for asset pricing and macro economics more broadly.

In this paper we carry out a fairly rigorous analysis of a dynamic general equilibrium model with many different industries, in which firms within each industry interact strategically in a product market. Specifically, we embed a dynamic oligopoly into a standard general equilibrium model with a continuum of production technologies and *CES* preferences. In contrast to the textbook “monopolistic competition” framework, in which monopolists in each industry compete across varieties of goods, we consider multiple producers in each industry and study the dynamics of intra-industry collusion. As a result, our setup is akin to “oligopolistic competition.” We integrate these industries into a standard consumption based asset pricing framework with a representative agent who consumes the goods and governs valuations of future (risky) cash flows, and hence influences the dynamics of intra-industry competition through continuation values.

Aggregate consumption in the economy is endogenously determined by the strategic considerations in each industry. Thus, the interplay between asset pricing and industrial organization is at the heart of this paper. The markups in an industry are determined endogenously, as a subgame perfect outcome in which firms weigh the value of high short-term profits that can be obtained by aggressive pricing, against the long-term profits that are obtained when firms cooperate. In addition to fundamental “taste/technology” shocks to the economy, firms’ profits, values and returns therefore vary with the changing competitive environment. The effects of strategic interaction is important in the sense that small technological shocks can have a drastic impact on the equilibrium outcome and on asset prices, through the mechanism of strategic competition.

Our paper makes three types of contributions. First, given that the setting with multiple industries and strategically interacting firms is fairly complex, we spend substantial effort on developing a rigorous understanding of the model, deriving general equilibrium existence results and exploring the properties of the model. Our main result in this part of the paper is Proposition 7 which shows the existence of equilibrium under minimal assumptions.

Second, we provide a qualitative characterization of the economy’s equilibrium and show that several interesting properties may arise. Specifically, the economy may have multiple qualitatively very different equilibria. Given a market price of risk, the equilibrium in each industry is unique, but there may be multiple prices of risk that support the same economy, leading to multiple equilibria.

We also illustrate how strategic competition can amplify the volatilities of real and financial variables far beyond any technological shocks, and small shocks to long term growth can have dramatic impact on the equilibrium. Short-term fluctuations of productivity, on the other hand,

do not matter at all.

Strategic competition may also generate Pareto inefficiencies. If all industries are monopolized or all industries are competitive, then there are no relative distortions in the economy and the representative agent’s consumption levels (which includes both wages and firm profits) are maximal. The intermediate case in which competitiveness varies across industry can be Pareto inefficient. There is therefore a role for policy in our framework.

Our third set of results is to develop asset pricing implications. The model relates industry- and economy wide-characteristics, and expected returns in the stock market. For the benchmark case with monopolistic firms, the results are quite straightforward. Under some technical conditions, firms in industries with pro-cyclical product demand have higher expected returns than firms in countercyclical industries. This is in line with the intuition that pro-cyclical firms generate value in good states of the world, in which marginal utility is low, and that such firms are therefore discounted at a higher discount rate. When there is strategic competition, however, this intuition breaks down, and non-monotonic relationships between product demand and expected returns may occur. The general implication is that industry characteristics (product demand, industry concentration and markups) should be informative about a firm’s expected returns in the stock market. We generate testable empirical implications along these lines.

Our paper is related to the Industrial Organization literature on strategic competition over the business cycle (see, e.g., Bagwell and Staiger, 1997). Our model takes this literature as a starting point, but extends the approach to allow for multiple industries and endogenous pricing of risk in an economy with risk averse agents. Our paper is also related to the extensive business cycle literature (Kydland and Prescott, 1982; Long and Plosser, 1983; Gabaix, 2011; Acemoglu, Carvalho, Ozdaglar, and Tahbaz-Salehi, 2011). Specifically, our model highlights how strategic interaction between firms (as an alternative to technological shocks) can generate endogenous business cycle fluctuations. It may also lead to inefficient outcomes, suggesting a role for policy makers. Our paper is further related to a small, but growing finance literature that explores the relationship between firms’ strategic environment in the product market and their expected returns in the stock market (Hou and Robinson, 2006; Hoberg and Phillips, 2010; Lyandres and Watanabe, 2010), which find that industry concentration is informative about a firm’s expected returns. Hou and Robinson (2006) document that firms in highly concentrated industries earn lower returns. Hoberg and Phillips (2010) find stronger “boom-bust” cycles in competitive industries, in that historically high valuations in competitive industries are stronger predictors of negative abnormal returns than in more concentrated industries. Lyandres and Watanabe (2010) suggest that cross sectional variations in industry concentration can explain the size and value premiums. These effects are all consistent with our model. Finally, our paper is related to the literature that use additional balance and income sheet items, beyond proxies for free cash flows, to explain expected returns (see, e.g., the discussion in Novy-Marx (2010)). Our model provides an explanation, based on first principles, for why such additional items matter.

To get the many different pieces of the model to fit together in a consistent and tractable framework, we have been forced to think carefully about modeling choices. To have a strategic trade-off among firms between competition and cooperation, multiple periods are needed, and an infinite horizon economy turned out to be the most tractable. Given the intricacies of strategic games in continuous time, a discrete approach turned out to be superior to a continuous time approach, especially since the increased tractability of the continuous time setting came at the

cost of *ad hoc* assumptions needed about the profitability of firms after off the equilibrium path moves. By choosing a discrete state spaces with time invariant Markov transition processes we were able to use the powerful analytical tools available for such processes. Finally, having a continuum of industries, each of which containing a finite number of firms, allowed us to assume that each firm takes the market price of risk as given, while still generating aggregate general equilibrium effects of firms' strategic behavior.

The rest of the paper is organized as follows. In Sections 2 and 3, we introduce the model and derive some properties that must hold in equilibrium. In Section 4, we study industry equilibrium, i.e., the equilibrium that is obtained under strategic competition in a specific industry, given an exogenously specified price of risk. In Section 5, we endogenize the price of risk, and derive the main results for existence and properties of general equilibrium. In Section 6, we study asset pricing implications of the model. Finally, concluding remarks are made in Section 7. All proofs are delegated to the Appendix.

2 Model Framework

Consider an infinite horizon, discrete time, discrete state economy in which time is indexed by $t \in \mathbb{Z}_+$ and the time t state of the world is denoted by $s_t \in \{1, 2, \dots, S\}$.¹ In each of a continuum of industries, a discrete number of strategic firms, subject to productivity shocks, produce and sell a unique non-storable consumption good. A representative agent supplies all labor, owns all firms and consumes all goods. We will mainly be interested in time invariant economies, in which $p(z, t_1) = p(z, t_2)$ if $s_{t_1} = s_{t_2}$. In such economies, we employ the shorthand notation, $p_s(z)$, and similarly for all other variables of interest. Going forward, the subscript s is thus used when studying time invariant outcomes, whereas the the argument t is used in the more general (possible time dependent) analysis.

There is a continuum of linear production technologies, indexed by $z \in [0, 1]$, each of which permits workers to produce a unique consumption good, also indexed by z . Such goods cannot be stored, but must be consumed in the period in which they are produced. An industry comprises $N(z) \geq 1$ identical firms, each of which uses the same technology. Here $N(z)$ is an integer.

There is uncertainty associated with the production technologies: To produce one unit of good z in state s , at time t , $\frac{l_s(z)}{(1+g)^t}$ units of human capital are needed, where $l_s(z) = \frac{1}{A_s(z)}$. Here, $A_s(z)$ can be interpreted as a productivity shock around the long-term productivity growth path, and $g \geq 0$ represents the long-term productivity growth rate. Without loss of generality, we may assume that $g = 0$, since the general case, $g > 0$, can be obtained by scaling other variables in the model. As we shall see, it will be particularly easy to incorporate g into agents' preferences. Notice that the shock is indexed by the state and is industry-specific. For tractability we assume that $l : S \times [0, 1] \rightarrow \mathbb{R}_{++}$ is a function that satisfies standard integrability conditions so that aggregation across industries is possible.

Each period there is a transition between states which is governed by a Markov process with

¹Here, $\mathbb{Z}_+ = \{0, 1, \dots\}$ is the set of non-negative integers. Also, we follow the standard convention that \mathbb{R}_+ is the set of nonnegative real numbers, whereas \mathbb{R}_{++} is the set of strictly positive real numbers.

time invariant transition probabilities:

$$\mathbb{P}(s_{t+1} = j | s_t = i) = \Phi_{i,j}. \quad (1)$$

Here, $\Phi_{i,j}$ refers to the element on the i th row and j th column of the matrix $\Phi \in \mathbb{R}_+^{S \times S}$. We assume that Φ is irreducible and aperiodic, so that the process has a unique long-term stationary distribution. With a slight abuse of notation, we will sometimes write S , for the set $1, \dots, S$, when there can be no confusion.

Labor is supplied by a representative agent, who in each period divides her one unit of human capital across all the industries and earns a state-contingent wage, w_s . Demand for each good is also determined by her preferences, which are represented by Dixit-Stiglitz *CES* preferences with elasticity of substitution $\theta > 1$,

$$\tilde{C}(t) = \left(\int_0^1 c(z, t)^{\frac{\theta-1}{\theta}} dz \right)^{\frac{\theta}{\theta-1}}. \quad (2)$$

We note in passing that preferences with a more general state dependent utility specification are also covered by our specification.²

In addition to her wage income, the representative agent owns shares in all firms which pay out state contingent industry profits, $\pi_s(z)$, as dividends.³ Her total income, y_s , is then defined as:

$$y_s = w_s + \int_0^1 \pi_s(z) dz, \quad (3)$$

where state-contingent profits satisfy

$$\pi_s(z) = [p_s(z) - w_s l_s(z)] c_s(z). \quad (4)$$

Let $p(z, t)$ denote the price of good z at time t . For tractability, we define the *gross markup function* $Q_s(z) = \frac{p_s(z)}{w_s l_s(z)}$. This is determined as the equilibrium of an infinite horizon repeated game with perfect public information. The timing of the stage game in each period, t , is as follows. First, the state, s_t is revealed. Then firms simultaneously announce their gross markups, Q . The markup of firm i in industry z at time t is Q_t^i . Consumers demand the product from the producer with the lowest markup. If multiple firms announce the same Q , total demand in sector z is evenly shared between these firms. The firms then go out and hire workers to meet demand.

Firms condition their action at time t on the entire history of past actions and states up to time t . The relevant history of each industry z , h_t is defined as the entire sequence of markups and states:

$$h_t = \left\{ \left\{ Q_\tau^i \right\}_{i=1}^N, s_\tau \right\}_{\tau=1}^t, \quad (5)$$

with h_0 representing the empty history. Thus, a time- t , industry- z strategy for firm i is a mapping from $h_{t-1} \times S$ to a chosen markup, Q_τ^i , $f : h_{t-1} \times S \rightarrow R_{++} \in R_{++}^{h_{t-1} \times S} = F_t^i$. Here,

²Consider the more general $\tilde{C}(t) = \left(\int_0^1 v_{s_t}(z) c(z, t)^{\frac{\theta-1}{\theta}} dz \right)^{\frac{\theta}{\theta-1}}$. Here, the state dependent ‘‘taste’’ function $v_s(z)$ can easily be reduced to the case where $v_s(z) \equiv 1$, by transforming the productivity, $A_s(z) \mapsto v_s(z)^{(\theta-1)/\theta} A_s(z)$. Such a transformation can be interpreted as a numeraire change, where the amount of a unit of goods is redefined in each state. A state dependent taste function could, for example, represent an agent’s higher utility of an umbrella in a rainy state than in a sunny state of the world.

³Note, that w_s is a free variable and can be normalized to any arbitrary constant.

the second parameter, $s \in S$, represents time t information about the state, which is available for the firm. A strategy for firm i is a sequence of time- τ strategies, $\{f_\tau^i\}_{\tau=1}^t$.

For any arbitrary price, from the agent's first order conditions, we can determine the state contingent demand for good z , $c_s(z)$ where $c : S \times [0, 1] \rightarrow \mathbb{R}_+$. Indeed, the good- z consumption demand of the representative agent with budget constraint $y(t) = \int_0^1 p(z, t)c(z, t)dz$ and facing a price function $p(z, t)$ is:

$$c(z, t) = \frac{y(t)}{p(z, t)^\theta P(t)^{1-\theta}}, \quad (6)$$

where $P(t) \equiv \left(\int_0^1 p(z, t)^{1-\theta} dz \right)^{\frac{1}{1-\theta}}$ can be interpreted as an appropriate price index (see derivation in Appendix A).

The *indirect* utility, $C(t)$, under the optimal consumption plan $c(z, t)$ is usually interpreted as the consumption aggregator of the economy and satisfies: $C(t) = \frac{y(t)}{P(t)}$.⁴ The level of $C(t)$ can be interpreted as the aggregate level of consumption at time t (see van Binsbergen, 2007). By normalizing $P(t)$ to 1 (without loss of generality), income, $y(t)$, is measured in real terms and can be meaningfully compared across periods.

Intertemporal decisions and risk aversion are governed by CRRA preferences over aggregate consumption with risk aversion parameter γ and subjective discount factor $\hat{\delta}$, i.e.:

$$U = E \left[\sum_{t=0}^{\infty} \hat{\delta}^t \frac{C(t)^{1-\gamma}}{1-\gamma} \right]. \quad (7)$$

In what follows it will be useful to consider the polar cases of the monopolist markup and that in perfect competition. If an industry is served by a monopolist, (so that $N(z) = 1$) he maximizes industry profits (see equation 4) subject to consumer demand (see equation 6) which leads to an optimal price of:

$$p_s^m(z) = \frac{\theta}{\theta-1} w_s l_s(z). \quad (8)$$

The markup is therefore $\frac{\theta}{\theta-1}$. If, on the other hand, $N(z)$ is infinite, then we expect prices to be set competitively, and $p_s^c(z) = w_s l_s(z)$. In this case, the markup is 1.

If the number of firms is finite but greater than one, we expect prices to be somewhere in between the competitive and monopolistic prices. Further, if $p \in [p^c, p^m]$ then $Q \in \left[1, \frac{\theta}{\theta-1} \right]$.⁵ Given Q , the supply of each industry is determined by the demand function (see equation 6).

The economy's exogenous parameters are summarized by the tuple $\mathcal{E} = (A, N, g, \Phi, \theta, \gamma, \hat{\delta})$. Here, the first four parameters describe the production environment whereas the last three describe consumer preferences. From the utility function (7), it follows that growth can be incorporated into an adjusted discount rate, i.e., that $\delta \stackrel{\text{def}}{=} (1+g)^{1-\gamma} \hat{\delta}$, summarizes all relevant information about growth, g , and the personal discount rate, $\hat{\delta}$.

⁴This is obtained by substituting demand (equation 6) into the utility function (equation 2) and using the definition of the price index P_s .

⁵Although the condition that $Q \in \left[1, \frac{\theta}{\theta-1} \right]$ arises naturally, for the results in this section it is not crucial, and we need only assume the weaker condition that Q is strictly positive for all z and s .

In what follows, we will sometimes decompose productivity shocks into the functions $\alpha_s(z)$ and \bar{A}_s where $\alpha : S \times [0, 1]$ and the vector $\bar{A} \in \mathbb{R}_+^S$. Specifically,

$$\alpha_s(z) \equiv \frac{A_s(z)^{\theta-1}}{\int_0^1 A_s(z)^{\theta-1} dz} = \left(\frac{A_s(z)}{\bar{A}_s} \right)^{\theta-1}, \quad \text{where} \quad (9)$$

$$\bar{A}_s \equiv \left[\int_0^1 A_s(z)^{\theta-1} dz \right]^{\frac{1}{\theta-1}}. \quad (10)$$

Here, \bar{A} represents the (non-linear) average productivity shock to the economy and $\alpha_s(z)$ captures the industry productivity shock relative to the economy. In other words, changes in $\alpha(z)$ across states are *idiosyncratic* shocks to individual industries, whereas changes in \bar{A} are *systematic* shocks. As a result of the normalization, the average relative industry state is equal to one, i.e., $\int_0^1 \alpha_s(z) dz = 1$. We note that instead of specifying A , we can equivalently specify the function of idiosyncratic shocks, α , and the vector of systematic shocks, $\bar{A} \in \mathbb{R}_{++}^S$. We will therefore alternatively use the representation $\mathcal{E} = (\alpha, \bar{A}, N, \Phi, \theta, \gamma, \delta)$ for exogenous variables in the economy.

While we are interested in general equilibrium; some partial equilibrium characterizations are of interest. First, for an arbitrary distribution of industry markups, we characterize aggregate consumption. This is important because it allows us to determine the endogenous discount factor which we derive from the preferences of the representative agent. We can then use this to value firms. We then fix the markups of all other industries and consider the partial equilibrium dynamics of one particular industry. That is, we characterize the strategic price setting behavior in one industry.

3 Aggregate Consumption and Pareto Efficiency for Arbitrary Markups

Aggregate consumption is an important endogenous variable. If the gross markup function, in each industry is specified for each state, i.e., $Q_s(z)$ is known, then together with the functions, $\alpha_s(z)$ and \bar{A}_s , the real outcome in the economy or the consumer's consumption bundle is completely determined, state-by-state. We will use the aggregate consumption in two ways. First, as a measure of inefficiency and second to price claims on financial assets. The fact that our economy can exhibit Pareto inefficiency distinguishes our paper from a standard real business cycle model. Our inefficiency is generated by firms' value-maximizing, strategic price setting behavior and the attendant distortion in labor allocation.

First, in order to characterize aggregate consumption it will be useful to define various notions of average markups. Formally, we define the (state dependent) p -th order power mean of Q as:

$$M_p(Q_s) = \left(\int \alpha_s(z) Q_s(z)^p dz \right)^{\frac{1}{p}}. \quad (11)$$

Notice that $\int_0^1 \alpha_s(z) dz = 1$ and so we interpret α as a probability measure where each industry obtains a weight according to its relative industry state. To simplify terms, we also introduce

the following notation:

$$\bar{Q}_s = M_{1-\theta}(Q_s), \quad (12)$$

$$e_s = \left(\frac{M_{-\theta}(Q_s)}{M_{1-\theta}(Q_s)} \right)^\theta \leq 1. \quad (13)$$

The $1 - \theta$ order power mean of markups, which is simply the harmonic mean for $\theta = 2$, plays a crucial role in our analysis. The ratio of the two power means, e_s , is bounded above by 1 due to Jensen's inequality and can be interpreted as a measure of efficiency; as the following proposition reveals.

Proposition 1. *The functions Q_s , α_s and \bar{A}_s define the industry equilibrium in the economy. Aggregate consumption, C_s , real income y_s , in state s are given by:*

$$C_s = y_s = \bar{A}_s e_s. \quad (14)$$

The fraction of real income that is derived from labor income is given by:

$$\omega_s = \frac{1}{e_s \bar{Q}_s}. \quad (15)$$

Real firm profits in sector z are:

$$\pi_s(z) = C_s \alpha_s(z) \bar{Q}_s^{\theta-1} \frac{Q_s(z) - 1}{Q_s(z)^\theta}. \quad (16)$$

From equation 14, aggregate consumption only depends on the aggregate shock \bar{A}_s and the ratio of average markups, $M_{-\theta}(Q)$ and $M_{1-\theta}(Q)$ embedded in e_s . Since $e_s \leq 1$, the upper bound of aggregate consumption, i.e., potential output, is given by the aggregate shock \bar{A}_s .

If all industries in the economy are perfectly competitive, so that $Q_s(z) \equiv 1$, then it is easy to see that the outcome is Pareto efficient (i.e., $C_s = \bar{A}_s$). On reflection, it follows that any economy in which markups across industries are the same in each state (i.e., $Q_s(z) \equiv k_s$ for all z and s) will also be Pareto efficient. A special case of this is an economy in which all industries are monopolized. In this case, the labor allocation is not distorted and all monopoly rents accrue to the representative agent and so consumption is maximal. We note that the efficiency of the completely monopolized outcome depends on our assumption that labor is the only factor input in the production function, which together with the fact that labor is always fully utilized implies that the relative allocation of labor across industries is what matters. In the fully monopolized case, markups are the same so no relative distortions exist across industries, hence the result. In this respect, the result is highly model dependent. However, the main points — which will also be valid in a more general setting — are that higher markups in some industries, in general equilibrium, do not necessarily lead to lower efficiency, and that the degree of markup *variation* across industries may be especially important in determining efficiency losses.

The economically interesting case is one in which some industries have high markups and some low. In such economies, the distortions, or alternatively the welfare costs, are the highest. Indeed, the greater the variance of markups across goods, the greater the wedge between $M_{-\theta}(Q_s)$ and $M_{1-\theta}(Q_s)$, and as such the smaller e_s . Welfare losses stem from the distorted

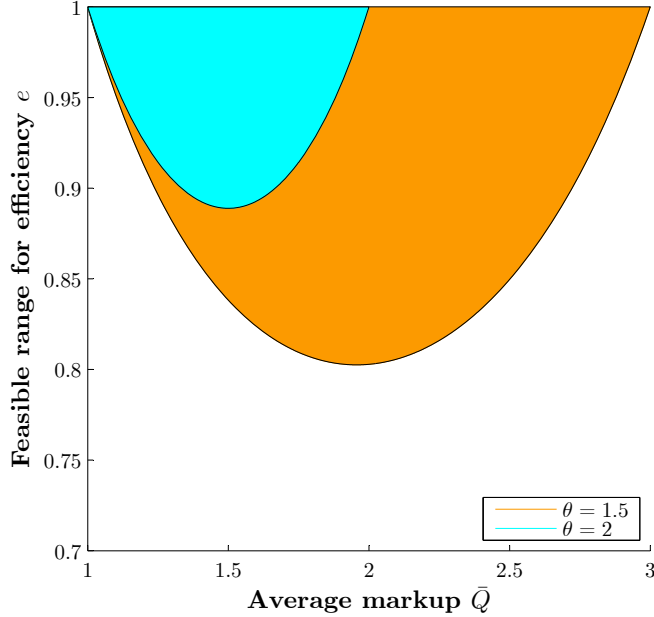


Figure 1. This graph plots the feasible range for efficiency e given an average markup of \bar{Q} in the economy. In the monopolistic economy, i.e., when $\bar{Q} = \frac{\theta}{\theta-1}$, and the competitive economy, i.e., $\bar{Q} = 1$, there are no welfare losses. In the intermediate region, potentially large welfare losses may occur if some industries charge high markups and some low markups. These potential welfare losses are larger for smaller θ , i.e., for higher monopoly markups of $\frac{\theta}{\theta-1}$.

allocation of labor to industries. Figure 1 illustrates the range of feasible welfare losses under the natural restriction that firms charge markups between 1 and $\frac{\theta}{\theta-1}$.

The fact that differences in markups across industries lead to inefficiency distinguishes our paper from those in the real business cycle literature. In papers such as Kydland and Prescott (1982) and Long and Plosser (1983), the outcome is efficient. Fluctuations in real variables are driven by technological shocks. Even though in our model, fluctuations in aggregate consumption are driven by labor productivity shocks, if markups are not constant across industries in each state, then the outcome is inefficient. Thus, our model shares the properties of business cycle models that shocks are real and technology driven and that prices adjust instantaneously to such shocks in equilibrium, but differ from standard business cycle models in that the outcome is not always efficient and in that there may therefore be a role for government intervention.

Aggregate consumption is also fundamental to pricing securities that are based on the real cash flows generated by firms. Such valuations will be pivotal to our understanding of industries strategic price setting behavior. To value any claim, we assume that there is a complete market of Arrow-Debreu securities in zero net supply traded, in addition to the stocks of the firms. As a result, the unique one period stochastic discount factor, SDF , satisfies:

$$SDF_{t+1} = \delta \left(\frac{C(t+1)}{C(t)} \right)^{-\gamma}. \quad (17)$$

This is the textbook stochastic discount factor for power utility, the only exception being that $C(t)$ now represents a properly defined consumption bundle (as opposed to a single consumption

good).

For time-invariant economies, since time- t profits of a firm depend only on the state, s , the information about the firm's future profits can be summarized in an S -vector, π , where π_s is the profit in state s . We can then define the ex-dividend value vector V as an S -vector where the element V_s represents the value of the firm in state s

$$V = \delta \Lambda_m^{-1} \Phi \Lambda_m (\pi + V). \quad (18)$$

Here, Λ_m is a diagonal matrix, with $m_s = C_s^{-\gamma}$ as its s th diagonal element. This argument leads to the following convenient formula for the value vector of the firm:

$$V = \Theta \pi, \quad (19)$$

where

$$\Theta = \Lambda_m^{-1} (I - \delta \Phi)^{-1} \Lambda_m - I. \quad (20)$$

The valuation operator Θ has strictly positive elements. This simply represents the fact that higher profits in some state s increase the present value of future profits, $V_{s'}$, in all states $s' = 1, \dots, S$.⁶

4 Partial Equilibrium in an Arbitrary Industry

Characterizing strategic price setting behavior in one industry is the first step towards endogenizing Q . We therefore characterize, as a function of industry and aggregate characteristics, when firms in a specific industry behave competitively, when a monopolistic outcome can be sustained, and when the outcome is neither of these extremes. In this section, because we focus on the dynamics of one specific industry, given the behavior of all other industries, we do not write out the z dependence of variables, but emphasize here that the following industry dynamics will occur in each industry, z .

As we have already observed, if an industry is monopolized, the outcome markup is $Q^m = \frac{\theta}{\theta-1}$ and when it is perfectly competitive the markup is $Q^c = 1$. Of course, in the latter case profits are zero, whereas in the former, monopolist profits, π_s^m are

$$\pi_s^m = \frac{[\theta - 1]^{\theta-1}}{\theta^\theta} \bar{Q}_s^{\theta-1} C_s \alpha_s \stackrel{\text{def}}{=} q_s C_s \alpha_s, \quad (21)$$

where $q_s = \frac{[\theta-1]^{\theta-1}}{\theta^\theta} \bar{Q}_s^{\theta-1}$.

Here, recall that C_s is the total consumption in state s , which is the same in all industries and is therefore systematic, whereas α_s is the productivity in the specific industry, which varies with z and is therefore idiosyncratic. Finally, $q_s \in \left[\frac{[\theta-1]^{\theta-1}}{\theta^\theta}, \frac{1}{\theta} \right]$ is a measure of aggregate market power. It attains $\frac{1}{\theta}$ if all sectors behave monopolistically and $\frac{[\theta-1]^{\theta-1}}{\theta^\theta} < \frac{1}{\theta}$ if the economy is perfectly competitive. Since goods are substitutes, real firm profits in a certain sector z are higher if other industries also charge higher markups, i.e., the aggregate market power, q_s is

⁶Recall that Φ is irreducible, so each state will be reached with positive probability, regardless of the initial state.

high. The corresponding value vector which gives the enterprise value in each state of the world is simply $V^m = \Theta\pi^m$.

Price setting is strategic in the intermediate case, in which $1 < N < \infty$. Following Abreu (1988), we are interested in industry equilibria that generate the highest industry profits sustainable by credible threats. We focus our attention on *symmetric, pure strategy subgame perfect* equilibria. The entire set of subgame perfect equilibria can be enforced with the threat of the worst possible subgame perfect equilibrium. As is standard in these games, the most severe punishment strategy is given by the perfectly competitive outcome. In this sub-game perfect equilibrium, industry profits, π_s , are zero in every state s .

If this is the punishment strategy, then any subgame perfect equilibrium must satisfy the following incentive constraints for each state s ,

$$\frac{\pi_s + V_s}{N} \geq \pi_s. \quad (22)$$

That is, the share of discounted present value of profits under collusion, $\frac{\pi_s + V_s}{N}$, must be greater or equal to the best-possible one period deviation of capturing the entire industry demand π_s and zero profits afterwards.⁷

In the maximum profit equilibria, in each state, s , firms in an industry choose the vector of state contingent markups to maximize the value function, V_s , given the value in each of the other states of the world, V_{-s} , subject to incentive compatibility (equation 22),

$$V_s = \arg \max_{Q_s} V_s | V_{-s}, \quad (23)$$

for all s . Here, Q_s maps to V_s via (16,19). Obviously, the case $N = 1$ is trivial: It leads to profits of π^m . We therefore focus on the case when $N \geq 2$.

In principle (23) is a complex optimization problem. First, strategic firms need to solve a state dependent infinite horizon state problem, second the optimal price is a highly non-linear function of profits. However, within our model's setting, finding the solution is actually quite straightforward. First, we note that the dynamic equilibrium can be viewed as a linear programming problem in which firms choose profits instead of prices, replacing Q_s in (23) with π_s . In the unique solution, either the industry splits the monopoly profits or the incentive constraints bind. Thus, viewed as an optimization problem over profits, the problem is linear. Second, the specific form of this corresponding linear programming problem asserts that the solution is the same for each state, and the optimization therefore collapses to a static, state independent, linear programming problem. This follows from the following technical result:

Proposition 2. *Consider a strictly positive vector $\pi^m \in \mathbb{R}_{++}^S$, a strictly positive matrix $\Theta \in \mathbb{R}_{++}^{S \times S}$, and scalar $n \in \mathbb{R}_{++}$. Then there is a unique $\xi \in \mathbb{R}_+^S$ so that for all strictly positive $b \in \mathbb{R}_{++}^S$,*

$$\begin{aligned} \xi &= \arg \max_x b^T x, \quad s.t., \\ x &\leq \pi^m, \\ 0 &\leq (\Theta - nI) x. \end{aligned}$$

⁷We are implicitly assuming that firms can coordinate within an industry to achieve this best outcome with this equilibrium selection mechanism. We do *not*, however, assume that firms can coordinate across industries, since in a large economy there are many industries and global coordination therefore typically is not possible.

For each s , the solution has either the first or the second constraint binding, i.e., for each s , $\xi_s = \pi_s^m$ or $n\xi_s = \Theta\xi_s$.

To see that this technical result reduces (23) to a state independent problem, consider $b = \Theta^T \delta_s$ where δ_s is the vector of zeros, except for the s th element which is equal to one, $\delta_s = (0, \dots, 0, 1, 0, \dots, 0)$, and $n = N - 1$. The second constraint in the Proposition is then the same as (22), and the implication of the Proposition is therefore that the same solution to the relaxed problem $V_s = \arg \max_{\pi_s} V_s$, where the consistency of strategies in other states is now not considered, is optimal in each state of the world, s . Thus, the unique solution to (23), is given by markups that correspond to the profit vector $\pi = \xi$ (through (16)), where ξ is the unique solution defined in Proposition 2.

Going forward, it will be important to understand when the incentive constraint binds. This is because, as we have observed, Pareto inefficiencies arise if markups differ across industries. If the first constraint binds, then monopoly profits (which by definition are maximal) are achieved and the entire industry behaves like a monopolist in this state s . To measure the ‘‘tightness’’ of the monopolistic incentive constraint, we introduce the ‘‘tightness’’ vector, Γ , with element s denoting the s -state ratio of the present value of industry profits under monopoly markups to monopoly profits:

$$\Gamma_s = \frac{\pi_s^m + V_s^m}{\pi_s^m} = 1 + \frac{V_s^m}{\pi_s^m}. \quad (24)$$

If $\Gamma_{s_1} > \Gamma_{s_2}$ the incentive to deviate in state s_1 is smaller than in state s_2 , i.e., the present value of collusion is high relative to current period profits.

Lemma 1. *The tightness vector satisfies:*

$$\Gamma = (\Lambda_\kappa^{-1}(I - \delta\Phi)^{-1}\Lambda_\kappa)\mathbf{1}, \quad (25)$$

where $\Lambda_\kappa = \text{diag}(\kappa)$, and the vector κ has elements:

$$\kappa_s = \pi_s^m m_s = q_s C_s^{1-\gamma} \alpha_s.$$

Note that Γ can be calculated without knowledge of the actual equilibrium outcome π .⁸ Moreover, since N does not depend on the state, Γ is independent of N . The variable κ_s captures an important determinant of the incentive to cheat in a certain state, Γ_s . It consists of the state component of the industry profit, $q_s C_s \alpha_s$, weighted by marginal utility in state s , $m_s = C_s^{-\gamma}$. Inspection of the incentive constraint (see equation 22) also implies that Γ_s is fundamentally related to the number of firms in an industry that can sustain monopoly markups in all states.

Proposition 3. *Define*

$$N^m \stackrel{\text{def}}{=} \min_s (\Gamma_s). \quad (26)$$

Then, if $N \leq N^m$, monopolistic profits are sustainable in all states of the world, i.e., $Q = \frac{\theta}{\theta-1}$, $\pi = \pi^m$.

⁸The matrix $\Lambda_\mu^{-1}(I - \delta\Phi)^{-1}\Lambda_\mu$ is very reminiscent of the valuation operator Θ (see definition in equation 20). Thus, the value of Γ can be interpreted as the (cum-dividend) value of a perpetuity of 1, in a hypothetical world in which the marginal utility in state s is given by μ_s rather than m_s .

Thus, N^m determines the threshold number of firms above which it is not possible to sustain monopolistic rents in all states.

We next determine the opposite extreme, a threshold value of number of firms in an industry, N^c , above which perfect competition in all states prevail. We have:

Proposition 4. *The threshold number of firms for perfect competition in all states of the world, N^c , is given by:*

$$N^c = \frac{1}{1 - \delta}. \quad (27)$$

In any industry with $N = N^c$ the equilibrium profits are:

$$\pi_s = m_s^{-1} \min_j \kappa_j = C_s^\gamma \min_j \kappa_j. \quad (28)$$

In any industry in which $N > N^c$, the competitive outcomes with zero profits in each state, $Q \equiv 1$, $\pi \equiv 0$, is obtained.

We note that, N^c is entirely determined by economy-wide variables and does not even depend on the aggregate state. Further, at N^c , the incentive constraint is characterized by the indifference condition of a risk-neutral firm that compares the shared perpetuity value under collusion, $\frac{\pi}{1-\delta} \frac{1}{N^c}$, and the best possible one-period deviation, π .

We have thus characterized the solution for industries with $N \leq N^m$ firms and for industries with $N \geq N^c$ firms. What remains is to characterize the solution for industries with $N^m < N < N^c$ firms. For a special case, the solution is especially simple. We have:

Proposition 5. *If $\kappa_s = k$ for all s and some arbitrary constant k , monopoly profits are incentive compatible for all $N \leq N^c$. As a result, $\Gamma_s = N^c$ for all s , so that $N^m = N^c$. Equilibrium markups are:*

$$Q_s = \begin{cases} \frac{\theta}{\theta-1} & \text{for } N \leq N^c, \\ 1 & \text{for } N > N^c. \end{cases} \quad (29)$$

An immediate implication of this Proposition is that markups in such industries are never state dependent (regardless of the number of firms in the industry), since they are neither state dependent in the monopolistic case, nor in the competitive case.

It remains to characterize the industry equilibrium when $N^m < N < N^c$ in the general case. We have:

Proposition 6. *For an industry in which $N^m < N < N^c$,*

1. *Equilibrium profits, π_s are nonincreasing in N for each s , as are markups.*
2. *Equilibrium profits, π_s , are nondecreasing in $\alpha_{s'}$, for each s, s' , as are markups.*
3. *Equilibrium profits in all states are strictly greater than zero, $\pi_s > 0$ for all s (equivalently, markups are strictly greater than one).*

4. *There will be at least one state in which monopolistic profits are obtained, $\pi_s = \pi_s^m$ for some s .*
5. *Equilibrium profits (and markups) depend continuously on all parameters (N , C , Φ , α , and \bar{A}).*

Thus, given that the aggregate variables of the economy are known, the qualitative behavior of markups in different states of the world in a specific industry is well understood. Observe that each industry is small compared with the aggregate economy, and that firms in industry z take the dynamics of all other industries as exogenously given, i.e., they take Q as exogenously defined for all $z' \neq z$. This is rational, since, as a result of Proposition 1, aggregate consumption is not affected by an individual industry's behavior, and neither is then the pricing kernel.

We illustrate our findings with an example. Assume that aggregate consumption satisfies $C = (1, 2, 4)^T$ and $\alpha = (\frac{1}{2}, \frac{3}{4}, 1)^T$. Moreover, assume preference parameters of $\delta = 8/9$, $\gamma = 2$, and $\theta = 2$. It is easy to show that the tightness vector in this example satisfies:

$$\Gamma = (7, 9, 13)^T \tag{30}$$

Thus, monopoly markups are sustainable for $N \leq N_M = 7$. Given δ , the number of firms necessary to induce the competitive outcome is $N_C = 9$. Figure 2 plots state-contingent markups as a function of the number of firms. This example features pro-cyclical markups, i.e., $Q_1 \leq Q_2 \leq Q_3$. While profits are unusually high in the good state of the world (increasing the incentive to cheat), this effect is overwhelmed by the valuation effect, i.e., future profits are discounted at a lower rate in times of low marginal utility. In contrast, in the worst state of the world the incentive to cheat is exacerbated by high marginal utility.

5 General Equilibrium

5.1 Existence

We now show existence of general equilibrium and provide some fundamental characterization results. Recall that the economy is characterized by the tuple \mathcal{E} , i.e., by the real variables $\alpha : S \times [0, 1] \rightarrow \mathbb{R}_+$, $N : [0, 1] \rightarrow \mathbb{N}_+$, $g \geq 0$, $\bar{A} \in \mathbb{R}_{++}^S$, the irreducible aperiodic stochastic matrix, $\Phi \in \mathbb{R}_{++}^{S \times S}$, and the preference parameters, γ , θ , and $\hat{\delta}$. An equilibrium is represented by the markup function, $Q : S \times [0, 1] \rightarrow [1, \frac{\theta}{\theta-1}]$, which can then be used to calculate all other variables, e.g., aggregate competitiveness, q_s , industry monopolistic profits, $\pi_s^m(z)$, and realized industry profits $\pi_s(z)$. We assume that N and α are Lebesgue measurable functions, and impose the following technical condition:

Condition 1. *For all s , for almost all z , $c_0 \leq \alpha_s(z) \leq c_1$ for constants, $0 < c_0 \leq c_1 < \infty$.*

Before showing general existence, we discuss some invariance results which will be helpful in the proof. We first note that the following result follows immediately from Proposition 2:

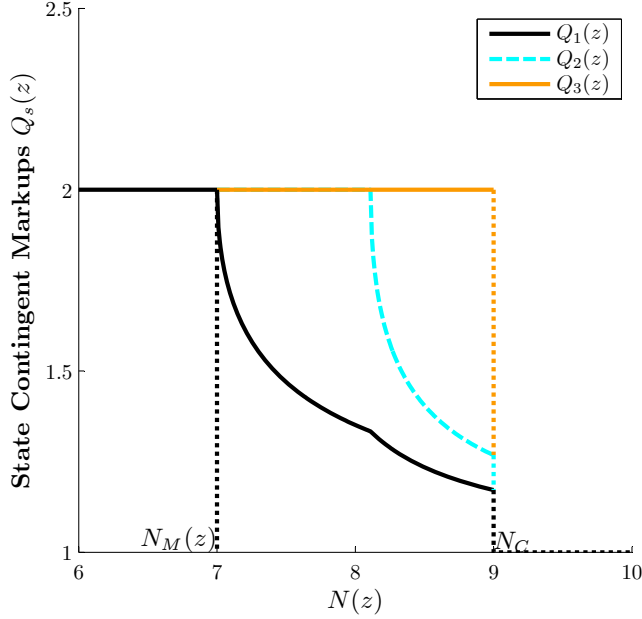


Figure 2. This graph plots the state contingent markups of one particular industry given aggregate consumption of $C = (1, 2, 4)^T$ and the relative industry state of $\alpha(z) = (2, 3, 4)^T$. If there are fewer than 7 firms in the industry, monopoly markups are sustainable in all states. Increasing the number of firms further cause the incentive constraint in state 1 to bind first, then in state 2 and finally, at $N_C = 9$, all markups collapse discontinuously to the competitive outcome, i.e., 1.

Lemma 2. *In any general equilibrium, any two industries with the same N and α have the same markups, Q , and profits, π .*

Also, we observe that it is only the distributional properties of N and α that are important for the aggregate characteristics of an equilibrium. This should come as no surprise given that the aggregate variables that are important for industry equilibrium only depend on the distributions. To be specific, we define the (cumulative) distribution function $F : \mathbb{N}_+ \times [c, C]^S \rightarrow [0, 1]$, where $F(n, s_1, \dots, s_S) = \lambda(\{(z : N(z) \leq n \wedge \alpha_1(z) \leq s_1 \cdots \wedge \alpha_S(z) \leq s_S\})$, and λ denotes Lebesgue measure. Thus, $F(n, \alpha_1, \dots, \alpha_S)$ denotes the fraction of industries with number of firms less than or equal to n , and productivities $\alpha_s(z) \leq \alpha_s$ for all s . We then have

Lemma 3. *Given two economies, 1 and 2, that are identical except for that their functions determining number of firms and productivity, N and α , differ. Assume that the economies have the same distribution function, F . Then they have the same equilibria in the sense that for each equilibrium in the first economy, there is an equilibrium in the second, such that any two industries, z and z' in the first and second economy, respectively, for which $N^1(z) = N^2(z')$ and $\alpha_s^1(z) = \alpha_s^2(z')$ for all s , have the same industry markups in each state of the world, $Q_s^1(z) = Q_s^2(z')$ for all s .*

We now have the following general existence result:

Proposition 7. *General equilibrium exists in any economy that satisfies Condition 1.*

Thus, only the technical conditions of integrability and boundedness of productivity functions across industries is needed to ensure the existence of equilibrium. We note that Proposition 7 makes no claim about equilibrium uniqueness — a subject that will be explored later in this section.

5.2 Equilibrium Competitiveness

Given that general equilibrium exists, our first objective is to understand how competitiveness in the economy depends on its parameters, \mathcal{E} . One way of measuring competitiveness is to study how large the mass of industries is that are perfectly competitive:

Definition 1. *The competitiveness, k , of an equilibrium is defined as the mass of industries that are perfectly competitive,*

$$k = \lambda(\{z : Q_s(z) = 1, \forall s\}).$$

We recall from the previous analysis that an industry is either competitive in all states or in no state, so requiring perfect competition in all states is in no way restrictive. We now have

Proposition 8. *The competitiveness of an equilibrium depends on δ and the function N , but not on Φ , α , \bar{A} or θ .*

Thus, somewhat surprisingly, the only real parameter that is important for competitiveness is the long-term growth rate, g . Recall that $\delta = \hat{\delta}(1+g)^{1-\gamma}$. If $\gamma < 1$, a higher long-term growth rate leads to higher competitiveness, i.e., more industries in which there is perfect competition. If $\gamma > 1$ on the other hand, a higher long-term growth rate leads to lower competitiveness. Compared with the risk neutral setting ($\gamma = 0$), competitiveness is higher in the economy with risk averse agents as long as the economy is growing in the long term, $g > 0$ (since $\delta(1+g)^{1-\gamma} < \delta(1+g)$). This is unsurprising, since future growth is valued less by an agent with a concave utility function, and it is therefore more tempting for firms to deviate from a cooperative outcome when agents have such preferences. Interestingly, shorter term fluctuations, represented by Φ , *never* matter for the competitiveness. Neither does the distribution of productivity over industries, α , nor aggregate productivity \bar{A} .

In fact, since, given δ and N , all equations that define the equilibrium (65-69) are smooth in all arguments, we typically expect all equilibrium properties to depend smoothly on all other parameters.⁹ Thus, small comparative static changes in these variables typically have limited effect on the equilibrium. In contrast, even very small changes in δ , e.g., driven by small changes in the long-term growth rate, g , or risk aversion, γ , or in the function, N , may have drastic effects on the equilibrium outcome.

⁹This is the “typical” behavior of such a set of smooth functions, although, for specific parameter choices it may not be the case — for example when there are multiple solutions, and some solutions only exist for a smaller set of parameter values.

5.3 An Example

The powerful implications of our general equilibrium analysis are best shown with the following workhorse example. Consider an economy with $S = 3$ states and 10 distinct industries represented by the sets I_j ,

$$I_j = \left\{ z : z \in \left[\frac{j-1}{10}, \frac{j}{10} \right] \right\}, \quad j = 1, \dots, 10.$$

With a slight abuse of terminology, we will call the I_j sets “industries,” although each set represents many identical industries. There is no exogenous aggregate volatility, i.e., the aggregate productivities A_s satisfy:

$$\bar{A}_1 = \bar{A}_2 = \bar{A}_3 = 1.$$

The relative industry productivity for each distinct industry j are defined as in Table 1.

	I_1	I_2	I_3	I_4	I_5	I_6	I_7	I_8	I_9	I_{10}
α_1	1	0.6	0.2	0.4	1.6	1.8	1.5	1.1	1.3	0.5
α_2	0.6	0.8	1.2	1.3	0.7	1.3	1.1	1.3	1.2	0.5
α_3	1.65	1.53	1.13	1.26	0.5	0.25	0.25	1.35	1.35	0.73

Table 1. Productivity in different industries and states.

We impose the following structure on the transition matrix Φ , using $p = 0.2$.

$$\Phi = \begin{bmatrix} 1-p & p & \dots & & & & & & & & \\ & p & 1-2p & p & \dots & & & & & & \\ & & p & 1-2p & p & \dots & & & & & \\ & & & & \dots & & & & & & \\ & & & & & & p & 1-p & & & \end{bmatrix}, \quad (31)$$

i.e., $\Phi_{1,1} = 1 - p$, $\Phi_{1,2} = p$, $\Phi_{i,i-1} = p$, $\Phi_{i,i} = 1 - 2p$, $\Phi_{i,i+1} = p$, $i = 2, \dots, S - 1$, $\Phi_{S,S} = 1 - p$, $\Phi_{S,S-1} = p$, where $0 < p < 0.5$. As we will discuss in Section 6 this particular structure satisfies useful properties.

The preference parameters satisfy $\theta = \frac{6}{5}$, $\gamma = 3$ and $\delta = \frac{9}{10}$.

5.3.1 Amplification of Business Cycles

It follows from Proposition 1 and the subsequent discussion that the above described economy features no aggregate uncertainty in the monopolistic economy, i.e., $N(z) = 10$ or if the competitive outcome obtains, i.e., $N(z) > 10$. Thus, these two benchmark cases imply:

$$C_1 = C_2 = C_3 = 1. \quad (32)$$

To show how strategic considerations can cause business cycles, i.e., create endogenous volatility of aggregate output, assume that $N(z) = 8$, for all z . Then the choice of markups shown in Table 2 constitutes an equilibrium outcome generating aggregate consumption of

	I_1	I_2	I_3	I_4	I_5	I_6	I_7	I_8	I_9	I_{10}
Q_1	6	6	6	6	1.89	3.59	4.52	6	6	6
Q_2	6	6	1.74	2.07	6	6	6	6	6	6
Q_3	6	6	6	6	6	6	6	6	6	6

Table 2. Markups in different industries and states.

$$C = (0.89, 0.83, 1)^T, \quad (33)$$

which also represents efficiency e for this specification. In this example, the strategic behavior of firms does not only create deviations from potential output of 1 (level effects), but is also the *only* source of aggregate business cycle fluctuations. In the best state of the economy, i.e., $s = 3$, all firms charge the monopoly markup of $\frac{\theta}{\theta-1} = 6$, as the incentive to deviate is low ($\gamma > 1$). Industries 1, 2, and 8 – 10 charge the monopoly markup across all states whereas the remaining industries charge low markups in the relatively bad states 1 and 2.

5.3.2 Uniqueness

Our existence result makes no claims with regards to uniqueness.¹⁰ Given an economy, \mathcal{E} , there may be multiple equilibria whenever a nonzero measure of firms fails the condition of perfect competition. The parametrization above reveals that there is indeed another equilibrium to the previous example supported by a different choice of markups, as shown in Table 3: Now,

	I_1	I_2	I_3	I_4	I_5	I_6	I_7	I_8	I_9	I_{10}
Q_1	6	6	6	6	1.54	1.76	1.83	6	6	6
Q_2	6	6	6	6	6	6	6	6	6	6
Q_3	1.94	1.87	4.72	6	6	6	6	6	6	6

Table 3. Markups in different industries and states for alternative equilibrium.

aggregate consumption in this equilibrium satisfies:

$$C = (0.8, 1, 0.84)^T.$$

Again, business cycles are endogenous. However, despite the same technology specification, the second equilibrium is very different from the first one. The worst state in the previous example, i.e., state 2, now achieves potential output of 1 as all industries charge monopolistic markups. While industries 5 – 10 charge qualitatively similar markups compared to the first equilibrium, the incentive constraints for industries 1 – 4 bind in different states. Multiplicity of equilibria implies that coordination *across* industries becomes relevant. In contrast, collusion within an industry produces a unique outcome given the behavior of all other industries and the implied stochastic discount factor (see Proposition 2).

¹⁰The proof is based on Schauderer’s fixed point theorem, which has little to say with respect to uniqueness.

5.3.3 Equilibrium Sensitivity

We also want to highlight that small changes in the growth rate or risk aversion may have large welfare implications by taking the economy from a Pareto efficient, perfectly competitive, outcome to one in which some industries are competitive and others are not.

Continuing with our example above, assume that all industries have 10 firms and that $\delta = 0.899$. In this case, equilibrium is unique and given by the perfectly competitive outcome. Now, consider a small change of δ to 0.9. Suddenly, collusion becomes feasible and the outcome depends on the endogenous choice of Q . As a result, aggregate consumption decreases from $C = (1, 1, 1)^T$ to

$$C = (0.80, 0.82, 0.86)^T. \quad (34)$$

The previous results highlight that although our model is based on a similar framework as real business cycle models (Kydland and Prescott (1982), Long and Plosser (1983)), endogenous variations of aggregate variables over the business cycle may arise. One implication of these results is that significant business cycle fluctuations may arise even when aggregate “technological” shocks are small. A recent strand of literature has aimed at explaining how technological shocks at the individual firm or industry level do not diversify out, but may affect aggregate productivity. Gabaix (2011) notes that if the distribution of firm size is heavy-tailed firm-specific shocks may indeed affect aggregate productivity. In Acemoglu, Carvalho, Ozdaglar, and Tahbaz-Salehi (2011), it is suggested that intersectoral input-output linkages may lead to “cascades effects” where a shock in one industry spreads through the economy and thereby becomes an aggregate shock.

The mechanism in our model is quite different, more along the lines suggested in Jovanovic (1987), who shows that idiosyncratic shocks may not cancel out in strategic games with a large number of players. In our previous example, aggregate productivity is constant across states, but because it varies at the sectoral level, the strategic behavior of firms leads to aggregate consumption shocks. We believe that this potentially provides an important mechanism for understanding the sources of aggregate fluctuations in the economy.

As noted, the endogenous business cycle fluctuations and potential multiple equilibria suggests a role for policy makers. This is noteworthy because our model shares many properties with standard business cycle models, in which outcomes are efficient. For example, there are no standard frictional costs in our model. Of course, the assumption that the number of firms in each industry is exogenously determined could be viewed as equivalent to assuming high entry costs. However, even if there were no entry costs, the competitive outcome may not prevail. For example, the outcome with $N = 10$ firms in each industry and aggregate consumption (34) is an equilibrium in the economy with zero costs of entry, since profits would immediately drop to zero if another firm entered an industry, so no firm has an incentive to do so. Moreover, there are no bubbles in the model. Instead, our key deviation from the traditional approach is to assume that intraindustry prices are strategically determined, instead of being determined through a Walrasian mechanism.

6 Asset Pricing

For convenience, we rewrite the previously derived pricing formulas (19,20)

$$V = \Theta\pi,$$

where

$$\Theta = \Lambda_m^{-1}(I - \delta\Phi)^{-1}\Lambda_m - I.$$

Here, $\Lambda_m = \text{diag}(m)$, where $m_s = C_s^{-\gamma}$ is the representative agent's marginal utility in state s . Without loss of generality, we assume that $C_1 \leq C_2 \leq \dots \leq C_S$, so that $m_1 \geq m_2 \geq \dots \geq m_S$. Thus, low- s states represent recessions where consumption is lower than normal, whereas high- s states represent expansion periods with abnormally high consumption. In the special case of a risk-neutral representative agent, $m = 1$, the pricing formula reduces to

$$V = ((I - \delta\Phi)^{-1} - I)\pi \stackrel{\text{def}}{=} (\Psi - I)\pi,$$

where $\Psi = (I - \delta\Phi)^{-1}$ is the resolvent, which replaces Φ since payoffs are perpetual and discounted. We also define $\hat{\Psi} = \Psi - I = \sum_{i=1}^{\infty} \delta^i \Phi^i$.

A general property of consumption based equilibrium models is that assets that pay off in good states of the world are worth less — and thereby have a higher expected return — than assets that pay off in bad states of the world. We would therefore expect such a result to hold within our setting too.

6.1 Arrow-Debreu Perpetuities

We first study simple Arrow-Debreu “perpetuities” and show that, given some natural technical restrictions on the resolvent, Ψ (and thereby on the transition matrix, Φ), the property above holds within our setting too. Specifically, we study securities 1_j , $1 \leq j \leq S$, that pay off one unit at time $t = 1, 2, \dots$ if and only if the economy is in state j at time t . The (*ex dividend*) expected return on Arrow-Debreu perpetuity 1_j in state i is denoted μ_{ij} , and it follows from (19,20), that

$$\mu_{ij} = \frac{[\Phi\Lambda_m^{-1}\Psi\Lambda_m]_{ij}}{[\Lambda_m^{-1}\hat{\Psi}\Lambda_m]_{ij}}. \quad (35)$$

The reason why extra technical restrictions are needed is that in the general case a state that is instantaneously “good” (i.e., has a low m_s) may actually be quite “bad,” because it may be very likely that the economy switches to a bad state (a high m_s state) in the next period. Compare an asset that pays off in a good state with one that pays off in a slightly worse state, in such an economy. The first asset, although instantaneously paying off in a better state than the first may actually be viewed as a “bad-state” asset, given the likely short-term dynamics of the economy. This may change the ordering of expected returns of the two assets. To avoid such pathological situations, we need something like a monotone likelihood ratio property (MLRP) to rank the conditional probabilities across states. Following Karlin (1968), we therefore define

Definition 2. A matrix, B , is said to be totally positive of order 2 (TP2) if, for every $i < j$ and $k < \ell$, $B_{ik}B_{j\ell} - B_{jk}B_{i\ell} \geq 0$.

For discrete state spaces, total positivity of the transition matrix corresponds to the monotone likelihood ratio property.¹¹ The following proposition introduces conditions that ensure that the ordering of expected returns of Arrow-Debreu perpetuities is monotone.

Proposition 9. *Define μ_{ij} to be the expected return in state i on the Arrow-Debreu perpetuity 1_j . Assume that the resolvent, Ψ , satisfies the following two conditions:*

$$\Psi \text{ is TP2}, \tag{36}$$

$$\Psi_{i+1,i+1}\Psi_{ii} - \Psi_{i,i+1}\Psi_{i+1,i} \geq \max(\Psi_{i+1,i+1}, \Psi_{i,i}), \quad 1 \leq i \leq S - 1. \tag{37}$$

Then, for all i , for all j and $j' > j$,

$$\mu_{ij'} \geq \mu_{ij},$$

i.e., in each state of the world, the expected return on an Arrow-Debreu perpetuity is higher, the higher state of the world in which it makes its payments.

We note that the condition needed for the Proposition to be satisfied only involves Ψ , not any other parameters of the economy. Also, (37) imposes a stronger condition than total positivity of order 2 on the diagonal of Ψ , since nonnegativity is not sufficient for (37) to be satisfied. In the monopolistic case, $m_s = C_s^{-\gamma} = \bar{A}_s^{-\gamma}$, so aggregate productivity in the form of \bar{A} provides a strong ranking of expected returns: Arrow-Debreu perpetuities that pay off in high-productivity states have higher expected returns than Arrow-Debreu securities that pay off in low-productivity states, and this is true regardless of which state, s , the economy is in.

We expect the above intuition to hold for more general assets too. Intuitively, the payouts of any firm in a stationary equilibrium can be viewed as those of a portfolio of Arrow-Debreu perpetuities, and firms that have relatively higher productivity in good states will have portfolios that load up more on high-productivity Arrow-Debreu perpetuities, and therefore also have higher expected returns. We show that this is indeed the case, in Section 6.2. We stress, however, that this intuition crucially depends on the assumption of a monopolistic economy. Without this assumption, the result is invalid, as shown in Section 6.3.

It is easy to show that (36,37) are always satisfied in an economy with two states:

Corollary 1. *In any economy with two states, $S = 2$, (36,37) are satisfied.*

It is possible to get reversal of the results in an economy with three states, however.

Example: Consider the 3-state economy, with

$$\Phi = \begin{bmatrix} 0.5 & 0.1 & 0.4 \\ 0.1 & 0.8 & 0.1 \\ 0.1 & 0.1 & 0.8 \end{bmatrix},$$

¹¹The close relationship between MLRP and TP2 can be easiest seen from the MLRP of a parametric family of strictly positive, smooth, density functions, $f(x|y)$, being given by $\frac{\partial^2 \log(f(x|y))}{\partial x \partial y} \geq 0$. A Taylor expansion shows that $\frac{\partial^2 \log(f(x|y))}{\partial x \partial y} \approx \frac{1}{\Delta x \Delta y} \frac{1}{f(x|y)} (f(x + \Delta x|y + \Delta y)f(x|y) - f(x + \Delta x|y)f(x|y + \Delta y))$ for small Δx and Δy . The TP2 condition is the discrete version of the term within the parentheses, when $f(i\Delta x|j\Delta y)$ is replaced by the matrix elements B_{ij} .

and $\delta = 0.9$, leading to

$$\Psi \approx \begin{bmatrix} 2.9688 & 2.4324 & 4.5988 \\ 1.4063 & 5.1351 & 3.4586 \\ 1.4063 & 2.4324 & 6.1613 \end{bmatrix}.$$

Now, Ψ is not TP2, since, choosing $i = 2, k = 1, j = 3, l = 2$, in Definition 2, we get

$$\Psi_{2,1}\Psi_{3,2} - \Psi_{3,1}\Psi_{2,2} = -3.8007 < 0.$$

Thus, (36) is not satisfied. Given that the marginal utilities are $m_1 = 11, m_2 = 10, m_3 = 1$, it is straightforward to use (38) to get

$$\mu_{2,1} = 1.99, \quad \mu_{2,2} = 1.64, \quad \mu_{2,3} = 2.73,$$

so in state 2, the Arrow-Debreu perpetuity that pays off in state 2, 1_2 , has a lower expected returns than the perpetuity that pays off in state 1, 1_1 , although state 1 is worse than state 2. This is thus a counterexample to the ranking of expected returns.

It is clear why this nonmonotonicity arises. Although state 2 is slightly better than state 1, once state 1 is reached, there is a decent chance that the economy will quickly move to state 3, since $\Phi_{1,3} = 0.4$, whereas the economy is likely to stay in state 2 for a substantial time period. Therefore, although the marginal utility is higher in state 1 than in state 2, it is a “better” state when dynamics are taken into account.

The example shows that some structure on Φ is indeed needed to obtain an ordering of expected returns. Together with Corollary 1, it also shows that the intuition from the 2-state economy, which has mainly been studied in previous literature, does not in general hold in multi-state economies.

As mentioned, the example above is in some sense pathological since the transition matrix has the property that it may be better to be in a state that short-term is worse. We will therefore focus on economies that satisfy (36,37), going forward. An example of an S -state economy in which the conditions are satisfied is when Φ has the structure outlined in equation 31: i.e., $\Phi_{1,1} = 1 - p$, $\Phi_{1,2} = p$, $\Phi_{i,i-1} = p$, $\Phi_{i,i} = 1 - 2p$, $\Phi_{i,i+1} = p$, $i = 2, \dots, S - 1$, $\Phi_{S,S} = 1 - p$, $\Phi_{S,S-1} = p$, where $0 < p < 0.5$. In this case, we have

Proposition 10. *In an $S \geq 2$ state economy in which Φ is defined as in (31), conditions (36,37) are satisfied.*

We next study the implications of the previous results for expected returns of general firms. We first focus on the monopolistic case, and show that firms with pro-cyclical demand have higher expected returns than firms with countercyclical demand. We then show that no such ordering occurs in the oligopolistic case.

6.2 Monopolistic Economies

As in Section 3, we study a specific industry, z , but suppress the z dependence for notational convenience. We wish to understand the relationship between how industry product demand and productivity vary with the state of the economy, and the expected returns in that industry.

Demand and productivity are conveniently summarized in α . Our intuition is that the analysis carried out for Arrow-Debreu securities should generalize such that firms with relatively higher demand in good states have higher expected returns than firms with relatively higher demand in bad states.

Of course, given the duality between productivity and product demand in our model, in which both are summarized in the function $\alpha(z)$, the previous argument might as well have been given in terms of pro- and countercyclical *productivity*. In line with previous literature, we will use the “product demand” terminology.

The expected return on a general firm that pays $\pi \in \mathbb{R}_+^S$ (where π_s is the payment made in state s), given that the economy is in state i , is denoted by μ_i , and from (19,20), it follows that

$$\mu_i = \frac{\sum_j [\Phi \Lambda_m^{-1} \Psi \Lambda_m]_{ij} \pi_j}{\sum_j [\Lambda_m^{-1} \hat{\Psi} \Lambda_m]_{ik} \pi_k}. \quad (38)$$

Given that $\pi_s \propto C_s \alpha_s$, (see (910,21) in the monopolistic economy, we can then write μ as a function of product demand, A , $\mu_i = \mu_i(\alpha) = \mu_i(\alpha_1, \dots, \alpha_S)$.

The following Proposition confirms our intuition that firms with more pro-cyclical normalized demand — in the sense that normalized demand is higher in good states — have higher expected returns than firms with countercyclical normalized demand — in the sense that it is higher in bad states:

Proposition 11. *Consider an industry with demand (productivity) $\alpha \in \mathbb{R}_+^S$ in an economy that satisfies (36,37). The expected return in state i is μ_i . Then, for any state $1 \leq i \leq S$, there is a cut-off state, $1 \leq \bar{s} \leq S$, such that*

- $\frac{\partial \mu_i}{\partial \alpha_s} \geq 0$ for all s such that $s \geq \bar{s}$, and
- $\frac{\partial \mu_i}{\partial \alpha_s} \leq 0$ for all s such that $s \leq \bar{s}$.

Thus, in each state, i , there is a cutoff state, so that firms with (all else equal) higher product demand in states above the cutoff state have higher expected returns than firms with (all else equal) lower product demand in states above the cutoff state. Similarly, firms with higher product demand in states below the cutoff state have lower expected returns than firms with lower product demand in state below the cutoff state. An immediate corollary is that

Corollary 2. *Increasing product demand in state S (weakly) increases a firm’s expected return, whereas increasing product demand in state 1 (weakly) decreases a firm’s expected return.*

These results verify our intuition about the relationship between demand cyclicity and expected returns.

6.3 Oligopolistic Economies

In this section we show that, since monopolistic markups cannot be sustained in all states in the oligopolistic setting, the link between product demand cyclicity and expected returns

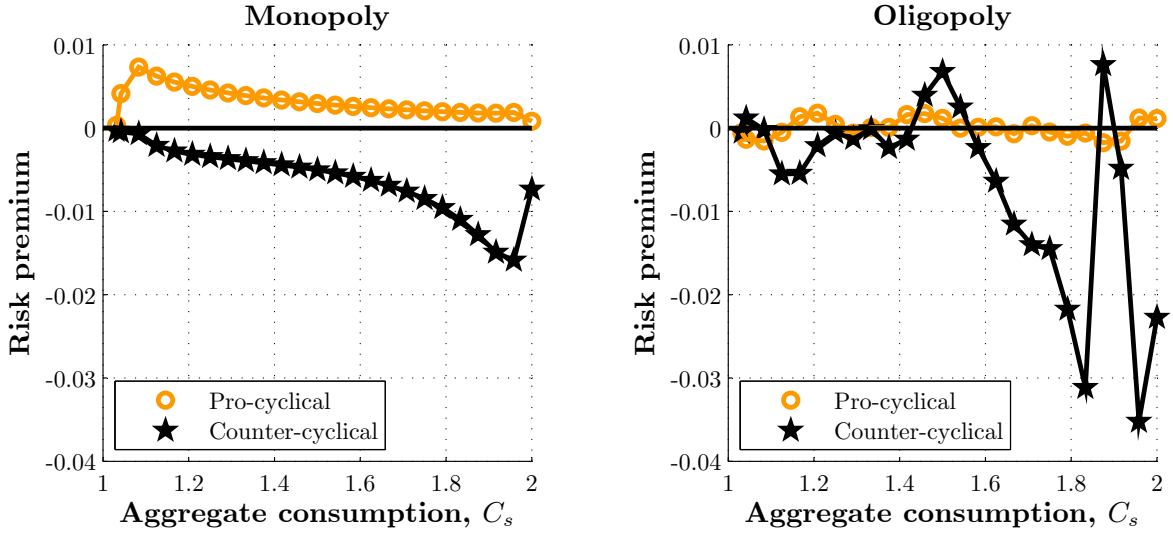


Figure 3. This graph plots the state-contingent risk premia for pro- and counter-cyclical industries under monopolistic behavior (left panel) and oligopolistic behavior (right panel).

measured by some type of profitability proxy, as well as proxies of product demand cyclicity, should be valuable conditioning variables.

To make this point explicit, we simulate 1,000 firms in different industries in our model to study how well firm and industry characteristics explain the cross section of expected returns. The example is necessarily very stylized, but captures the intuition of our paper. We focus on unconditional expected returns and use a similar specification as the one just studied. Again, we consider a 25-state economy with a monopolistic equilibrium (such that $\bar{A} \equiv C$), and focus on zero measure industries in which there may be strategic competition. The parameters in the economy are given by:

$$C_s = 1 + 0.4s, \quad 1 \leq s \leq S, \quad (41)$$

$\theta = 1.1, \gamma = 8$ and $\delta = 0.9$. Each firm, i , has productivity $\alpha_s = 1 + 9\zeta_{is}$, in state s , where ζ_{is} are uniformly $(0, 1)$ distributed and are independent across firms and states. Further, the number of firms in the industry in which firm i operates, $N_i = 6 + 4\zeta_i^N$, where ζ_i^N is uniformly $(0, 1)$ distributed, and independent across firms. Further N 's and α 's are jointly independent. The transition matrix between states is again tridiagonal, but we vary the probabilities across states, such that $\Phi_{i,i+1} = p_i^1, \Phi_{i,i-1} = p_i^2, \Phi_{i,i} = 1 - p_i^1 - p_i^2$. Thus, the structure of Φ is as before in that transitions only occur to neighboring states, but we allow the transition probabilities to vary with the state. The reason why we use a slightly more complex transition matrix is that the effects we wish to show are somewhat mitigated — although still present — in the special case of constant probabilities.¹² The coefficients p_i^1 and p_i^2 are given in the appendix.

We expect the number of firms in an industry, N_i , to be important for expected returns, because it will determine the competitiveness in that industry, and ultimately in which states of the world profits are generated. We also expect additional information to be captured

¹²Technically, constant probabilities leads to the one case in which the CAPM works quite well, because they imply a uniform stationary distribution.

Univariate				
Coefficient		a_1	a_2	a_3
		0.0056***	0.0034***	0.012***
R-square		0.12	0.59	0.60
Multivariate				
Coefficient		a_1	a_2	a_3
		-0.0017**	0.023***	0.024***
R-square	0.75			
Correlation				
	a_1	1	0.36	0.53
	a_2	0.36	1	0.62
	a_3	0.53	0.62	1

Table 4. Cross sectional regression of unconditional expected returns on market betas, number of firms, and the skewness of profits divided by productivity.

in the productivity and profits across states, α_{is} and π_{is} , respectively, and especially by the relationship between these two variables. One approach, which we will take, is to scale profits with productivity, and study the variable $\omega_{is} = \frac{\pi_{is}}{\alpha_{is}}$ across states for a given firm, i . We note that ω_{is} should in principle be straightforward to estimate in practice, since π_{is} is proxied by various profit measures on a firm's income sheet (e.g., by net income or NOPLAT), and α_{is} could be captured by estimating the product demand function in an industry. We have experimented with summary statistics of ω_i across states. It turns out that the mean of ω_i is highly correlated with N_i , since they both capture average competitiveness across states. Similarly, the volatility of ω_i across states also turns out to be highly correlated with N_i and thereby capturing similar information. The skewness of ω , however, is less correlated and — as it turns out — also captures significant information about the cross section of returns.¹³ We therefore define $\Omega_i = skew(\omega_{is})$.

We regress unconditional expected returns on unconditional market betas, β_i , number of firms in the industry, N_i , and skewness of profits over productivity, Ω_i , across firms,

$$\mu_i = a_i^1 \beta_i + a_i^2 N_i + a_i^3 \Omega_i + \epsilon_i, \quad i = 1, \dots, 1,000,$$

using an ordinary least square regression. We also carry out univariate regressions for each of the variables. The results are shown in Table 4. We see that in the univariate regressions, both N_i and Ω_i are superior in determining expected returns, with r-squares of about 0.6, compared with market beta which only has an r-square of 0.12. In the multivariate regressions, the coefficient on market beta is even negative. The total r-square in this regression reaches 0.75, and is only marginally lower when market beta is left out (about 0.73, not reported in the table). The correlation between beta and number of firms is quite low, about 0.36, whereas the other correlations are somewhat higher, between 0.5-0.6. Thus, in line with the intuition of our study, firm and industry characteristics together provide a good characterization of the cross section of expected returns, capturing nonredundant but somewhat overlapping pieces of information.

¹³It is outside of the scope of this study to investigate why this is the case.

7 Concluding Remarks

We have developed general equilibrium in a dynamic economy with a continuum of industries each of which comprises a finite number of firms. The framework is quite tractable, and the strategic interaction between firms in each industry is straightforward to characterize. We establish the existence of general equilibrium and establish dynamic properties of the economy including equilibrium markups and attendant asset prices.

The central premise of our model is that firms, maximizing shareholder value, are not always price takers but can be price setters. High prices in an industry can be sustained if firms value the future flow of profits over any immediate increases in market share garnered by undercutting. Of course, the rate at which future profits are discounted depends both on the representative agent's risk aversion and on the behavior of the aggregate economy.

The strategic behavior of firms leads to two distinct type of predictions. First, we develop various general equilibrium effects that can interpreted in light of the macro-economy. Even in an economy with no aggregate uncertainty, if the relative productivity of various industries changes, so does their ability to sustain collusive outcomes. These changes can affect both the level and the volatility of aggregate consumption; in short our model exhibits endogenous volatility. Second, we show how industry characteristics are related to asset prices. Specifically, industry characteristics (such as concentration and productivity) affect firms' ability to set prices strategically. An immediate implication of this is that these inter-industry differences should help to explain asset prices.

While conceptually simple, the model is sufficiently rich that it presents many avenues for future research, both on the strategic behavior of firms and on the general equilibrium properties. First, we take the number of firms in each industry as exogenous. However, a natural extension would be to model strategic entry and exit. We speculate that this would lead even more strategic volatility. Second, while we present various stylized examples, it would be natural to structurally calibrate or estimate the model. For example, it is worth noting that the consumption aggregator is highly non-linear. This means that a linear aggregator (such as a price weighted one) will consistently underestimate the volatility of the utility an agents enjoys from a consumption stream. It is an empirical question as to whether this framework provides new empirical insight into the relationship between the volatility of asset prices and aggregate consumption, however it is an interesting avenue for further research. Finally, our model has potentially new implications for policy actions. Indeed, the existence of multiple equilibria suggests that government intervention may be beneficial.

A Optimal Expenditure Share

Maximizing the objective function in state s given by equation 2 subject to the budget constraint results in the following Lagrangian:

$$L_s = \left(\int_0^1 c_s(z)^{\frac{\theta-1}{\theta}} dz \right)^{\frac{\theta}{\theta-1}} - \lambda \left(y_s - \int_0^1 p_s(z) c_s(z) dz \right). \quad (42)$$

The first-order conditions with regards to consumption for some arbitrary $c_s(z')$ and $c_s(z'')$ imply:

$$c_s(z')^{\frac{\theta-1}{\theta}-1} \left(\int_0^1 c_s(z)^{\frac{\theta-1}{\theta}} dz \right)^{\frac{\theta}{\theta-1}-1} = \lambda p_s(z'), \quad (43)$$

$$c_s(z'')^{\frac{\theta-1}{\theta}-1} \left(\int_0^1 c_s(z)^{\frac{\theta-1}{\theta}} dz \right)^{\frac{\theta}{\theta-1}-1} = \lambda p_s(z''). \quad (44)$$

Using these two equations results in:

$$\frac{\frac{1}{\theta} c_s(z')^{\frac{-1}{\theta}}}{\frac{1}{\theta} c_s(z'')^{\frac{-1}{\theta}}} = \frac{p_s(z')}{p_s(z'')}. \quad (45)$$

Since z' was arbitrary, this gives us:

$$c_s(z) = c_s(z'') \frac{p_s(z'')^\theta}{p_s(z)^\theta}. \quad (46)$$

Plugging this expression $c_s(z)$ into the budget equation yields:

$$y_s = \int_0^1 p_s(z) c_s(z'') \frac{p_s(z'')^\theta}{p_s(z)^\theta} dz. \quad (47)$$

Solving for $c_s(z'')$ gives us:

$$c_s(z'') = \frac{1}{p_s(z'')^\theta} \frac{y_s}{P_s^{1-\theta}}. \quad (48)$$

where

$$P_s = \left(\int_0^1 p_s(z)^{1-\theta} dz \right)^{\frac{1}{1-\theta}} \quad (49)$$

represents the appropriate price index for *CES* preferences.

B Proofs

Proof of Proposition 1

Since the wage rate is a free variable, we can always normalize it in such a way that $P_s = 1$. Taking $Q_s(z)$ as given: the wage rate is implicitly determined by

$$\left(\int_0^1 [w_s l_s(z) Q_s(z)]^{1-\theta} dz \right)^{\frac{1}{1-\theta}} = 1. \quad (50)$$

Solving for w_s yields:

$$w_s = \frac{1}{\left(\int_0^1 l_s(z)^{1-\theta} Q_s(z)^{1-\theta} dz\right)^{\frac{1}{1-\theta}}}. \quad (51)$$

Using the definition of $\alpha_s(z) = \frac{l_s(z)^{1-\theta}}{\bar{A}_s^{\theta-1}}$ (see equation 9) and the power mean definition we can write this as:

$$w_s = \frac{\bar{A}_s}{M_{1-\theta}(Q_s)}. \quad (52)$$

Plugging the demand function, $c_s(z)$ (see equation 6), into the profit equation (see equation 4), yields profits of:

$$\pi_s(z) = [p_s(z) - w_s l_s(z)] \frac{1}{p_s(z)^\theta} y_s. \quad (53)$$

Writing $p_s(z) = w_s l_s(z) Q_s(z)$ and using the definition of $\alpha_s(z) = \frac{l_s(z)^{1-\theta}}{\bar{A}_s^{\theta-1}}$ yields:

$$\pi_s(z) = w_s^{1-\theta} l_s(z)^{1-\theta} [Q_s(z) - 1] \frac{1}{Q_s(z)^\theta} y_s \quad (54)$$

$$= \alpha_s(z) \frac{Q_s(z) - 1}{Q_s(z)^\theta} \bar{A}_s^{\theta-1} y_s w_s^{1-\theta} \quad (55)$$

$$= \alpha_s(z) \frac{Q_s(z) - 1}{Q_s(z)^\theta} \frac{y_s}{[M_{1-\theta}(Q_s)]^{1-\theta}}, \quad (56)$$

where the last line uses the definition of $w_s = \frac{\bar{A}_s}{M_{1-\theta}(Q_s)}$. Total income satisfies:

$$y_s = w_s + \int \pi_s(z) dz \quad (57)$$

$$= w_s + y_s \int \alpha_s(z) \frac{Q_s(z) - 1}{Q_s(z)^\theta} \frac{1}{[M_{1-\theta}(Q_s)]^{1-\theta}} dz. \quad (58)$$

Solving for y_s yields (using w_s):

$$y_s = \frac{\bar{A}_s}{M_{1-\theta}(Q_s)} \frac{1}{\left(1 - \frac{1}{[M_{1-\theta}(Q_s)]^{1-\theta}} \int \alpha_s(z) \frac{Q_s(z)-1}{Q_s(z)^\theta} dz\right)} \quad (59)$$

$$= \frac{\bar{A}_s}{M_{1-\theta}(Q_s)} \frac{1}{\left(1 - \frac{1}{[M_{1-\theta}(Q_s)]^{1-\theta}} \left[\int \alpha_s(z) Q_s(z)^{1-\theta} dz - \int \alpha_s(z) Q_s(z)^{-\theta} dz\right]\right)}. \quad (60)$$

Employing the definition of power means results in:

$$y_s = \frac{\bar{A}_s}{M_{1-\theta}(Q_s)} \frac{1}{\left(1 - \frac{1}{[M_{1-\theta}(Q_s)]^{1-\theta}} \left[[M_{1-\theta}(Q_s)]^{1-\theta} - [M_{-\theta}(Q_s)]^{-\theta}\right]\right)} \quad (61)$$

$$= \frac{\bar{A}_s}{M_{1-\theta}(Q_s)} \frac{1}{\left(1 - \frac{[M_{1-\theta}(Q_s)]^{1-\theta}}{[M_{1-\theta}(Q_s)]^{1-\theta}} + \frac{[M_{-\theta}(Q_s)]^{-\theta}}{[M_{1-\theta}(Q_s)]^{1-\theta}}\right)} \quad (62)$$

$$= \frac{\bar{A}_s}{M_{1-\theta}(Q_s)} \frac{[M_{-\theta}(Q_s)]^\theta}{[M_{1-\theta}(Q_s)]^{\theta-1}} = \bar{A}_s \left[\frac{M_{-\theta}(Q_s)}{M_{1-\theta}(Q_s)}\right]^\theta. \quad (63)$$

Proof of Proposition 2

Let $x < y$ denote that $x \leq y$ and $x \neq y$. Also, define $z = x \vee y \in \mathbb{R}^S$, where $z_s = \max(x_s, y_s)$ for all s . Clearly, $x \leq x \vee y$, where the inequality is strict if there is an s such that $y_s > x_s$. Finally, define the set $K = \{x : 0 \leq x, x \leq \pi^*, nx \leq \Theta x\}$. Note that K is compact.

Now, there is a unique maximal element of K , that is, there is a unique $\xi \in K$, such that for all $x \in K$ such that $x \neq \xi$, $\xi > x$. This follows by contradiction, because assume that there are two distinct maximal elements, y and x , then clearly $z = x \vee y$ is strictly larger than both x and y . Now, it is straightforward to show that $z \in K$. The only condition that is not immediate is that $\Theta z \geq Nz$. However, this follows from $\Theta(x \vee y) \geq \Theta x \vee \Theta y \geq nx \vee ny = n(x \vee y) = nz$.

Now, since b is strictly positive, it is clear that ξ is indeed the unique solution to the optimization problem regardless of b . That one of the constraint is binding for each s also follows directly, because assume to the contrary that neither constraint is binding in some state s . Then ξ_s can be increased without violating either constraint in state s and, moreover, the constraints in all the other states will actually be relaxed, so such an increase is feasible. Further, since $b_s > 0$, it will also increase the objective function, contradicting the assumption that ξ is optimal.

Proof of Lemma 1

By definition: $V = \Lambda_\pi(\Gamma - \mathbf{1})$, so from (21), $\Lambda_\pi(\Gamma - \mathbf{1}) = \delta \Lambda_m^{-1} \Phi \Lambda_m (\pi + \Lambda_\pi(\Gamma - \mathbf{1}))$, leading to $\Gamma - \mathbf{1} = \delta \Lambda_\pi^{-1} \Lambda_m^{-1} \Phi \Lambda_m \Lambda_\pi \Gamma$. Now, observing (from (21)) that $\Lambda_\kappa = \Lambda_\pi \Lambda_m$, the result follows immediately.

Proof of Proposition 4

Let $n = N - 1$ and $K^*(n) \stackrel{\text{def}}{=} \{x : 0 \leq x, nx \leq \Theta x\}$. Now, $nx \leq (\Lambda_m^{-1}(I - \delta\Phi)^{-1}\Lambda_m - I)x$ is equivalent to $Ny \leq (I - \delta\Phi)^{-1}y$, where $y = \Lambda_m x \in \mathbb{R}_+^S$. We first show that $K^*(n) = \{0\}$ when $N > \frac{1}{1-\delta}$, which immediately implies that the only solution to the optimization problem in Lemma 2 is indeed the competitive outcome. Define the matrix norm $\|A\| = \sup_{x \in \mathbb{R}_+^S \setminus \{0\}} \frac{\|Ax\|}{\|x\|}$, where the l^1 vector norm $\|y\| = \sum_s |y_s|$ is used. Since Φ is a stochastic matrix, $\|\Phi^i\| = 1$ for all i and using standard norm inequalities it therefore follows immediately that

$$\|(I - \delta\Phi)^{-1}\| = \left\| \sum_0^\infty \delta^i \Phi^i \right\| \leq \sum_0^\infty \delta^i \|\Phi^i\| = \frac{1}{1-\delta},$$

and thus $\|(I - \delta\Phi)^{-1}y\| \leq \frac{1}{1-\delta}\|y\|$. Now, $Ny \leq (I - \delta\Phi)^{-1}y$ implies that $N\|y\| \leq \|(I - \delta\Phi)^{-1}y\|$, and therefore it must be the case that $N \leq \frac{1}{1-\delta}$, for the inequality to be satisfied for a non-zero y . Now, consider the case when $N = \frac{1}{1-\delta}$. Since $y = \mathbf{1}$ is an eigenvector to Φ with unit eigenvalue, it is also an eigenvector to $(I - \delta\Phi)^{-1}$ with corresponding eigenvalue $\frac{1}{1-\delta}$, leading to $x = \Lambda_m^{-1}\mathbf{1} = m^{-1}$. It is easy to show that this is the unique (up to multiplication) nonzero solution. Given the properties of Φ , the Perron-Frobenius theorem implies that this is indeed the *only* eigenvector with unit eigenvalue, and therefore also the only eigenvector to $(I - \delta\Phi)^{-1}$ with eigenvalue $\frac{1}{1-\delta}$. Now, take an arbitrary $y \in \mathbb{R}_+^S \setminus \{0\}$ as a candidate vector to satisfy the inequality, i.e., such that $z = (I - \delta\Phi)^{-1}y$ satisfies $z_i \geq Ny_i = \frac{1}{1-\delta}y_i$ for all i . Then, since $\|(I - \delta\Phi)^{-1}\| = \frac{1}{1-\delta}$, it follows that $\sum_i z_i \leq \frac{1}{1-\delta} \sum_i y_i$. The two inequalities can only be satisfied jointly if $z_i = \frac{1}{1-\delta}y_i$ for all i , and thus y is the already identified eigenvector. Thus, $K^*\left(\frac{1}{1-\delta}\right) = \{\iota m^{-1}, \iota \geq 0\}$. It follows immediately from the definition of the λ vector that the maximal ι that satisfies $\iota m_s^{-1} \leq \pi_s^* = q_s C_s \alpha_s$ for all s is $\min_s \lambda_s$, leading to the given form of the profit vector.

Proof of Proposition 5

If $\kappa_s = k$, the diagonal matrix Λ_κ becomes $\Lambda_\kappa = kI$ so that we obtain for Γ (see (25)):

$$\Gamma = (I - \delta\Phi)^{-1}\mathbf{1} = \frac{1}{1-\delta}\mathbf{1} = N^c\mathbf{1}. \quad (64)$$

This is because the eigenvalue of $(I - \delta\Phi)^{-1}$ associated with the eigenvector of $\mathbf{1}$ is given by $\frac{1}{1-\delta}$ (see Proof of Proposition 4). So, $N^m = \min_s(\Gamma_s) = N^c$.

Proof of Proposition 6

(1,2) follow from the definition of K in the proof of Lemma 2. It immediately follows that the set K is decreasing in N and increasing in each of α_s , which in turn immediately implies (1,2).

(3) follows from (1), and the fact that $\pi_s > 0$ for all s when the number of firms is N^c .

(4) follows from (1) and that $\pi_s = m_s^{-1}\pi_s^m m_s$ for the s that minimizes μ_s (see Proposition 3).

(5) follows from the fact that the objective function in Lemma 2 is a continuous function of all parameters and that (as long as N is strictly below N^c) the set K is compact, and depends continuously on all parameters, in the sense that if K and K' are defined for two sets of parameter values, then $D(K, K')$ approaches zero when the parameter values that define K' approach those that define K . Here, $D(K, K') = \sup_{x \in K'} \inf_{y \in K} |x - y|$.

Proof of Proposition 7

We wish to prove the proposition with a fixed point argument, and therefore define a fixed point relationship for the markup function, Q , which ensures that it defines an equilibrium. We define $R \stackrel{\text{def}}{=} \bar{N} \times [c, C]^S$, where $\bar{N} = \{1, 2, \lfloor N_c \rfloor + 1\}$, with elements $x = (n, \alpha_1, \dots, \alpha_S) \in R$. We will then work with functions $Q^0 : R \rightarrow [0, 1]^S$, and given such a function, the transformation to the standard markup function is given by $Q_s(z) = Q_s^0(\min(N(z), \lfloor N_c \rfloor + 1), \alpha_1(z), \dots, \alpha_S(z))$. The reason why we work with the canonical domain, R , rather than $S \times [0, 1]$, is that compactness properties needed for a fixed point argument are easier obtained in this domain. Given a function, $Q^0 : R \rightarrow [1, \frac{\theta}{\theta-1}]^S$, we define

$$p_s^0 = M_{-\theta}(Q_s) = \left(\int \alpha_s(z) Q_s(z)^{-\theta} dz \right)^{\frac{1}{-\theta}} = \left(\int_{x \in R} x_{s+1} Q^0(x)^{-\theta} dF(x) \right)^{\frac{1}{-\theta}}, \quad (65)$$

$$p_s^1 = M_{1-\theta}(Q_s) = \left(\int \alpha_s(z) Q_s(z)^{1-\theta} dz \right)^{\frac{1}{1-\theta}} = \left(\int_{x \in R} x_{s+1} Q^0(x)^{1-\theta} dF(x) \right)^{\frac{1}{1-\theta}}. \quad (66)$$

It follows immediately that the mapping from Q^0 to p_0 and p_1 is continuous (in L^1 topology) and since $\int \alpha(z) dz = 1$, that p_s^0 and p_s^1 lie in $[1, \theta/(\theta-1)]$. From (14), it follows that

$$C_s = \bar{A}_s \left(\frac{p_1}{p_0} \right)^\theta, \quad (67)$$

and from (21) that

$$\pi_s^m = \frac{1}{p_1^{1-\theta}} \frac{(\theta-1)^{\theta-1}}{\theta^\theta} \alpha_s C_s = \frac{1}{p_1^{1-\theta}} \frac{(\theta-1)^{\theta-1}}{\theta^\theta} x_{s+1} C_s. \quad (68)$$

Now, for each z , given $\pi^m \in \mathbb{R}_+^S$, the program in Lemma 2 provides a continuous mapping from π^m to

$$\pi_s \in \prod_1^S [0, \pi_s^m]. \quad (69)$$

We use (16) to define the operator \mathcal{F} , which operates on functions, and which is given by:

$$Q_s^1(x) = (\mathcal{F}(Q^0)(x))_s = 1 + \frac{p_1(s)^{1-\theta}}{C_s x_{s+1}} (Q_s^0(x))^\theta \pi_s.$$

Since each operation in (65-69) is continuous, it follows that \mathcal{F} is a continuous operator (in $L^1(\mathbb{R}^{1+S})$ -norm). Further, it also follows that if $Q_s^0(x) \in \left[1, \frac{\theta}{\theta-1}\right]$, then since $0 \leq \pi \leq \pi^m$, $1 \leq Q_s^1(x) \leq 1 + \frac{(\theta-1)^{\theta-1}}{\theta^\theta} (Q_s^0)^\theta \leq \frac{\theta}{\theta-1}$. Define, Z as the set of all functions, $Q : R \rightarrow [1, \theta/(\theta-1)]^S$, such that Q is nonincreasing in its first argument and nondecreasing in all other arguments. Then, from what we have just shown, together with Proposition 6, it follows that \mathcal{F} is a continuous operator that maps Z into itself. We also have

Lemma 4. *Z is convex and compact.*

We prove that the set, W , of nondecreasing functions $f : [0, 1] \rightarrow [0, 1]$, is convex and compact. The generalization to functions with arbitrary rectangular domains and ranges, $f : \prod_1^N [a_i, b_i] \rightarrow \prod_1^M [c_i, d_i]$, is straightforward, as is the generalization to functions that are nonincreasing in some coordinates and nondecreasing on others (as is Z). Convexity is immediate. For compactness, we show that every sequence of functions $f^n \in W$, $n = 1, 2, \dots$, has a subsequence that converges to an element in W . First, note that W is closed, since a converging (Cauchy) sequence of nondecreasing functions necessarily converges to a nondecreasing function. To show compactness, define the corresponding sequence of vectors $g^n \in [0, 1]^{2^j}$, for some $j \geq 1$, by $g_k^n = f_n(2^{-j}k)$, $k = 0, 1, \dots, 2^j - 1$. Now, since $[0, 1]^{2^j}$ is compact it follows that there is a subsequence of $\{f^n\}$, $\{f^{n_m}\}$ that converges at each point $2^{-j}k$, to some $g^* \in [0, 1]^{2^j}$. Define the function $h^j : [0, 1] \rightarrow [0, 1]$ by $h^j(x) = g_k^*$, for $2^{-j}k \leq x < 2^{-j}(k+1)$, which is obviously also in W . Next, take the sequence $\{f^{n_m}\}$, and use the same argument to find a subsequence that converges in each point $2^{-(j+1)}k$, $k = 0, \dots, 2^{j+1} - 1$, and the corresponding function $h^{j+1}(x)$. By repeating this step, we obtain a sequence of functions in W , h^j, h^{j+1}, \dots , such that for $m > j$,

$$\int_0^1 |h^m(x) - h^j(x)| dx \leq \sum_k (g_{k+1}^j - g_k^j) 2^{-j} \leq 2^{-j}.$$

Thus, h^j, h^{j+1}, \dots forms a Cauchy-sequence, which consequently converges to some function $h^* \in W$. Take a subsequence of the original sequence of functions, $\{f^{n_j}\}$, such that $\int |f^{n_j} - h^j| dx \leq 2^{-j}$. Then, for $m > j$, since

$$\begin{aligned} \int_0^1 |f^{n_m}(x) - f^{n_j}(x)| dx &= \int_0^1 |f^{n_m}(x) + h^m(x) - h^m(x) + h^j(x) - h^j(x) - f^{n_j}(x)| dx \\ &\leq \int_0^1 |f^{n_m}(x) - h^m(x)| dx + \int_0^1 |f^{n_j}(x) - h^j(x)| dx \\ &\quad + \int_0^1 |h^m(x) - h^j(x)| dx \\ &\leq 3 \times 2^{-j}, \end{aligned}$$

$\{f^{n_j}\}$ is also a Cauchy sequence and converges to $h^* \in W$. Thus, W is compact and the lemma is proved. Given Lemma 4 and the continuity of \mathcal{F} , a direct application of Schauder's fixed point theorem

implies that there is a $Q^* \in Z$, such that $\mathcal{F}(Q^*) = Q^*$. Now, given such a Q^* , and its associated π^m defined by (68), and given the functions, $N(z)$ and $\alpha_s(z)$, $0 \leq z \leq 1$, Lemma 2 can be used to construct $Q_s(z)$. Since Q and Q^* have the same distributional properties, and C , p_0 and p_1 , only depend on distributional properties, it immediately follows that Q constitutes an equilibrium. We are done.

Proof of Proposition 8

Follows immediately from Proposition 4.

Proof of Proposition 9

From (35), it follows that

$$\mu_{ij} = \frac{[\Phi \Lambda_m^{-1} \Psi \Lambda_m]_{ij}}{[\Lambda_m^{-1} (\Psi - I) \Lambda_m]_{ij}} = \frac{[\Phi + \Lambda_m^{-1} \hat{\Psi} \Lambda_m]_{ij}}{[\Lambda_m^{-1} \hat{\Psi} \Lambda_m]_{ij}} = \frac{\Phi_{ij} + \sum_k \Phi_{ik} \frac{1}{m_k} \hat{\Psi}_{kj} m_j}{\frac{1}{m_i} \hat{\Psi}_{ij} m_j}, \quad (70)$$

where $\hat{\Psi} = \Psi - I$. Now, since $\hat{\Psi} = \Psi - I = \delta\Phi + \delta^2\Phi^2 + \dots = \delta(\Phi + \Phi(\delta\Phi + \dots)) = \delta(\Phi + \Phi\hat{\Psi})$ it follows that

$$\frac{\Phi_{ij} + \sum_k \Phi_{ik} \hat{\Psi}_{kj}}{\hat{\Psi}_{ij}} = \frac{1}{\delta},$$

which we use to rewrite

$$\begin{aligned} \mu_{ij} &= m_i \left(\frac{\Phi_{ij} \frac{1}{m_i} + \sum_k \Phi_{ik} \frac{1}{m_i} \hat{\Psi}_{kj}}{\hat{\Psi}_{ij}} + \frac{\Phi_{ij} \left(\frac{1}{m_j} - \frac{1}{m_i} \right) + \sum_k \Phi_{ik} \left(\frac{1}{m_k} - \frac{1}{m_i} \right) \hat{\Psi}_{kj}}{\hat{\Psi}_{ij}} \right) \\ &= \frac{1}{\delta} + m_i \sum_k r_i(k) \Phi_{ik} \frac{\Psi_{kj}}{\hat{\Psi}_{ij}}, \end{aligned}$$

where $r_i(k) \stackrel{\text{def}}{=} \frac{1}{m_k} - \frac{1}{m_i}$ because of the ordering of the states is increasing in k for each i , such that $r_i(k) \leq 0$ if $k < i$, $r_i(k) \geq 0$ if $k > i$ and $r_i(i) = 0$. We now show that μ_{ij} is (weakly) increasing in j for each i , i.e., that the procyclicality ordering is indeed valid for the Arrow-Debreu securities. To see this, note that

$$\begin{aligned} \mu_{i,j+1} - \mu_{i,j} &\geq 0 \Leftrightarrow \sum_k r_i(k) \Phi_{ik} \left(\frac{\Psi_{k,j+1}}{\hat{\Psi}_{i,j+1}} - \frac{\Psi_{kj}}{\hat{\Psi}_{ij}} \right) \geq 0 \\ &\Leftrightarrow \sum_{k < i} r_i(k) \Phi_{ik} \left(\frac{\Psi_{k,j+1}}{\hat{\Psi}_{i,j+1}} - \frac{\Psi_{kj}}{\hat{\Psi}_{ij}} \right) + \sum_{k > i} r_i(k) \Phi_{ik} \left(\frac{\Psi_{k,j+1}}{\hat{\Psi}_{i,j+1}} - \frac{\Psi_{kj}}{\hat{\Psi}_{ij}} \right) \geq 0. \end{aligned}$$

Now, for $i < j$, $\hat{\Psi}_{ij} = \Psi_{ij}$, so the condition can be written

$$\sum_{k < i} r_i(k) \Phi_{ik} \left(\frac{\Psi_{k,j+1}}{\Psi_{i,j+1}} - \frac{\Psi_{kj}}{\Psi_{ij}} \right) + \sum_{k > i} r_i(k) \Phi_{ik} \left(\frac{\Psi_{k,j+1}}{\Psi_{i,j+1}} - \frac{\Psi_{kj}}{\Psi_{ij}} \right) \geq 0.$$

Now, since Ψ is TP2, $\frac{\Psi_{k,j+1}}{\Psi_{i,j+1}} - \frac{\Psi_{kj}}{\Psi_{ij}}$ is nonpositive for $k < i$ and nonnegative for $k > i$, so since $r_i(k)$ is nonpositive for the first term and nonnegative for the second term, the total expression is indeed weakly positive. An identical argument can be made for $i > j + 1$. For $i = j$, the expression becomes

$$\sum_{k < i} r_i(k) \Phi_{ik} \left(\frac{\Psi_{k,i+1}}{\Psi_{i,i+1}} - \frac{\Psi_{ki}}{\Psi_{ii} - 1} \right) + \sum_{k > i} r_i(k) \Phi_{ik} \left(\frac{\Psi_{k,i+1}}{\Psi_{i,i+1}} - \frac{\Psi_{ki}}{\Psi_{ii} - 1} \right) \geq 0.$$

Note that $\Psi_{ii} > 1$, $1 \leq i \leq S$, so all terms are positive. Since $\frac{\Psi_{ki}}{\Psi_{i,i+1}} \geq \frac{\Psi_{ki}}{\Psi_{ii}}$, the same argument as before is valid for the first term. For the second term, (37) implies that

$$\frac{\Psi_{i+1,i+1}}{\Psi_{i,i+1}} - \frac{\Psi_{i+1,i}}{\Psi_{i,i} - 1} \geq 0, \quad 1 \leq i \leq S - 1,$$

and thereby that the expression inside the parenthesis is nonnegative for $k = i + 1$. Further, since Ψ is TP2, $\frac{\Psi_{k,i+1}}{\Psi_{k,i}} \geq \frac{\Psi_{i+1,i+1}}{\Psi_{i+1,i}} \geq \frac{\Psi_{i,i+1}}{\Psi_{i,i} - 1}$ for $k > i + 1$, so $\frac{\Psi_{k,i+1}}{\Psi_{i,i+1}} \geq \frac{\Psi_{ki}}{\Psi_{ii} - 1}$ for all $k \geq i + 1$. Thus, the result follows. An identical argument shows that the result is also valid for $i = j + 1$, so the result indeed holds for all i . We are done.

Proof of Corollary 1

When $S = 2$, only one condition for (37), $i = 1$, needs to be checked. The general form of Φ is

$$\Phi = \begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix}, \quad 0 \leq p, q \leq 1,$$

leading to

$$\Psi = \frac{1}{(1-\delta(1-p))(1-\delta(1-q))-\delta^2 pq} \begin{bmatrix} 1-\delta(1-q) & \delta q \\ \delta p & 1-\delta(1-p) \end{bmatrix}.$$

The left hand side of (37) is now $\frac{1}{(1-\delta(1-p))(1-\delta(1-q))-\delta^2 pq}$, which is (weakly) greater than both Ψ_{11} and Ψ_{22} , so (37) is indeed satisfied. It is also greater than zero, which is the TP2 condition (36). We are done.

Proof of Proposition 10

The case $S = 2$ has already been covered. We therefore assume that $S \geq 3$. It follows that $R = \Psi^{-1} = (I - \delta\Phi)$ is a tridiagonal matrix with elements $R_{1,1} = R_{S,S} = z(1+q)$, $R_{i,i} = z(2+q)$, $i = 2, \dots, S-1$, and $R_{i,i+1} = R_{i+1,i} = -z$, $i = 1, \dots, S-1$, where $z = \delta p$, and $q = \frac{1-\delta}{\delta p}$. Now, following the argument in Pena (1995), since Ψ is a matrix with positive elements, it follows that R is a tridiagonal so-called M -matrix, and that $\Psi = R^{-1}$ therefore is a totally positive matrix (of any order). Thus, (36) is satisfied. To show (37), we use the representation of Ψ in Meek (1980):

$$\begin{aligned} \Psi_{i,j} &= e_{\min(i,j)} e_{S-\max(i,j)+1} + A_j e_{S-i+1} + B_j e_i, \\ e_i &= \frac{\sinh((i-1)\theta)}{\sinh((S-1)\theta)}, \\ \theta &= \operatorname{arcosh}\left(1 + \frac{q}{2}\right), \\ A_j &= \frac{1}{(1+q-e_{S-1})^2 - e_2^2} ((1+q-e_{S-1})e_{S-j+1} + e_2 e_j), \\ B_j &= \frac{1}{(1+q-e_{S-1})^2 - e_2^2} ((1+q-e_{S-1})e_j + e_2 e_{S-j+1}). \end{aligned}$$

Some algebra leads to the following expression

$$\begin{aligned} \Psi_{i+1,i+1} - \frac{\Psi_{i,i+1}\Psi_{i+1,i}}{\Psi_{i,i}} &= \frac{qe_2}{(1-\delta)\alpha} (e_{i+1} + e_{S-i}\beta)^2 \left(\frac{e_{S-i} + e_{i+1}\beta}{e_{i+1} + e_{S-i}\beta} - \frac{e_{S-i+1} + e_i\beta}{e_i + e_{S-i+1}\beta} \right), \text{ where} \\ \beta &= \frac{1+q-e_{N-1}}{e_2}, \quad \alpha = (1+q-e_{N-1})^2 - e_2^2. \end{aligned}$$

If this expression is (weakly) greater than one, then $\Psi_{i+1,i+1}\Psi_{i,i} - \Psi_{i,i+1}\Psi_{i+1,i} \geq \Psi_{i,i}$. Now, $\delta = \frac{1}{1+pq} \geq \frac{1}{1+0.5q}$, so a sufficient condition is that

$$T \stackrel{\text{def}}{=} \frac{(2+q)e_2}{\alpha} (e_{i+1} + e_{S-i}\beta)^2 \left(\frac{e_{S-i} + e_{i+1}\beta}{e_{i+1} + e_{S-i}\beta} - \frac{e_{S-i+1} + e_i\beta}{e_i + e_{S-i+1}\beta} \right) \geq 1.$$

Plugging in $q = 2(1 + \cosh(\theta))$, and simplifying, we get

$$T = \frac{2e^\theta (e^{2i\theta} + e^{(1+2S)\theta}) \cosh(\theta)}{e^{2i\theta} + e^{(3+2S)\theta}}.$$

Taking the derivative with respect to θ yields

$$\frac{\partial T}{\partial \theta} = -\frac{2(e^{(4+4S)\theta} - e^{(2+4i)\theta}) + (2S+1-2i)(e^{(5+2i+2S)\theta} - e^{(1+2i+2S)\theta})}{(e^{2i\theta} + e^{(3+2S)\theta})^2},$$

which by inspection is seen to be strictly negative. Thus, T is strictly decreasing in θ . Finally, an asymptotic expansion of T for large θ implies that

$$T = \frac{e^{4S\theta}(1 + o(\theta))}{e^{4S\theta}(1 + o(\theta))},$$

so $\lim_{\theta \rightarrow \infty} T(\theta) = 1$. Thus, $T(\theta) > 1$ for all θ , and therefore $\Psi_{i+1,i+1}\Psi_{i,i} - \Psi_{i,i+1}\Psi_{i+1,i} \geq \Psi_{i,i}$. An identical argument shows that $\Psi_{i+1,i+1}\Psi_{i,i} - \Psi_{i,i+1}\Psi_{i+1,i} \geq \Psi_{i+1,i+1}$, so (37) is satisfied. We are done.

Proof of Proposition 11

The proof is very similar to the proof of Proposition 9. From (38), it follows that

$$\mu_i = \frac{\sum_j \left[\Phi_{ij} + \sum_k \Phi_{ik} \frac{1}{m_k} \hat{\Psi}_{kj} m_j \right] \pi_j}{\sum_j \frac{1}{m_i} \hat{\Psi}_{ij} m_j \pi_j}, \quad (71)$$

where $\hat{\Psi} = \Psi - I$. Since $\hat{\Psi} = \Psi - I = \delta\Phi + \delta^2\Phi^2 + \dots = \delta(\Phi + \Phi(\delta\Phi + \dots)) = \delta(\Phi + \Phi\hat{\Psi})$ it follows that

$$\frac{\sum_j \left[\Phi_{ij} + \sum_k \Phi_{ik} \hat{\Psi}_{kj} \right] m_j \pi_j}{\sum_j \hat{\Psi}_{ij} m_j \pi_j} = \frac{1}{\delta},$$

which we use to rewrite

$$\mu_i = \frac{1}{\delta} + m_i \sum_k \left(r_i(k) \Phi_{ij} \frac{\sum_j \Psi_{kj} m_j \pi_j}{\sum_j \hat{\Psi}_{ij} m_j \pi_j} \right),$$

where, as before, $r_i(k) \stackrel{\text{def}}{=} \frac{1}{m_k} - \frac{1}{m_i}$, which defining $\xi_j \stackrel{\text{def}}{=} m_j \pi_j$, can be written

$$\mu_i = \frac{1}{\delta} + m_i \sum_k \left(r_i(k) \Phi_{ij} \frac{\sum_j \Psi_{kj} \xi_j}{\sum_j \hat{\Psi}_{ij} \xi_j} \right),$$

and since ξ_j is strictly increasing in α_j , we study

$$\frac{\partial \mu_i}{\partial \xi_n} = m_i \frac{\hat{\Psi}_{in}}{\sum_j \hat{\Psi}_{ij} \xi_j} \sum_k \left(r_i(k) \Phi_{ij} \left(\frac{\Psi_{kn}}{\hat{\Psi}_{in}} - \frac{\sum_j \Psi_{kj} \xi_j}{\sum_j \hat{\Psi}_{ij} \xi_j} \right) \right),$$

We define

$$F_n = \sum_{k < i} \left(r_i(k) \Phi_{ij} \left(\frac{\Psi_{kn}}{\hat{\Psi}_{in}} - \frac{\sum_j \Psi_{kj} \xi_j}{\sum_j \hat{\Psi}_{ij} \xi_j} \right) \right),$$

$$G_n = \sum_{k > i} \left(r_i(k) \Phi_{ij} \left(\frac{\Psi_{kn}}{\hat{\Psi}_{in}} - \frac{\sum_j \Psi_{kj} \xi_j}{\sum_j \hat{\Psi}_{ij} \xi_j} \right) \right),$$

to get

$$\frac{\partial \mu_i}{\partial \xi_n} = m_i \frac{\hat{\Psi}_{in}}{\sum_j \hat{\Psi}_{ij} \xi_j} (F_n + G_n).$$

A similar argument as in the proof of Proposition 9, shows that $F_{n+1} \leq F_n$, and $G_{n+1} \geq G_n$ ($n < S$) so it is indeed that case that if $\frac{\partial \mu_i}{\partial \xi_n} \geq 0$, then $\frac{\partial \mu_i}{\partial \xi_{n+1}} \geq 0$. Similarly, if $\frac{\partial \mu_i}{\partial \xi_n} \leq 0$ ($n > 1$), then $\frac{\partial \mu_i}{\partial \xi_{n-1}} \leq 0$.

It remains to show that both the case $\frac{\partial \mu_i}{\partial \xi_n} \geq 0$, and $\frac{\partial \mu_i}{\partial \xi_n} \leq 0$ occur, but this follows easily from the fact that

$$\min_j \frac{a_j}{b_j} \leq \frac{\sum_j a_j \xi_j}{\sum_j b_j \xi_j} \leq \max_j \frac{a_j}{b_j},$$

for arbitrary positive a_j ξ_j and strictly positive b_j . We are done.

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