Risk Sharing for the Long Run.
The gains from financial integration.*

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Abstract

Cole and Obstfeld (1991) pointed out that the welfare benefits of international portfolio diversification might be negligible. They obtain this result in the context of a model in which agents have time-additive constant relative risk aversion preferences. We revisit their conclusion by showing that a preference for the timing of the resolution of uncertainty combined with endowments containing a slowly moving trend can result in extremely high welfare gains. The model is also able to account for a large set of international finance stylized facts. Our setup allows us to bridge part of the gap between the current finance and international macroeconomic literatures.

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1 Introduction

There is an extensive economics and finance literature that addresses the potential benefits of financial liberalization. If on the one hand there is consensus on the significant inherent gain that may be attained in terms of portfolio diversification, decreased cost of equity capital, and reduced financing constraints, on the other hand, economics modeling has typically stumbled in the prediction of negligible welfare gains.[1]

Cole and Obstfeld (1991), for example, estimate that the incremental loss from a ban on international portfolio diversification is in the order of 0.20% of output per year. Tesar (1995) evaluates the utility gains from risk sharing for various market structure, country size, technology, and preferences, to conclude that the gains from risksharing range from zero to two percent of lifetime consumption. van Wincoop (1994) explores the possibility of non-time separability in preferences to conclude that countries like the United States and the United Kingdom would experience a permanent increase in lifetime consumption that is in most cases smaller than 1.5%. Gourinchas and Jeanne (2006) find that the welfare gain is equivalent to a 1% permanent increase in domestic consumption for the typical non-OECD country. The list of papers on this issue could go on, but the main finding would remain consistent.

Although all these models are able to accurately characterize the joint behavior of a large set of economic variables, they are typically silent about how closely they can track stock markets dynamics. Put differently, what are the welfare benefits of financial integration when one wants to explain simultaneously prices and quantities?

In order to answer this question, we propose a general equilibrium model that is able to simultaneously explain: (i) the degree of volatility of exchange rate fluctuations and of stochastic discount factors; and (ii) the volatility of net export and the amount of cross-country correlation and persistence of consumption growths. We show that in our setup opening the countries to international financial markets could result in benefits as large as 10% of lifetime consumption.

We assume that agents have risk-sensitive preferences in the sense of Hansen and Sargent (1995). This implies that investors have a preference for the timing of the

[1] See for example Bekaert, Harvey, and Lundblad (2005) for an extensive review of this literature.

resolution of uncertainty. We conduct our analysis for the case in which consumption is a Cobb-Douglas aggregation of domestic and foreign goods, both of which are tradable. Furthermore we let the dynamics of the growth rate of the endowments of the two countries be characterized by the presence of two slowly moving predictive factors. These components, denoted as long-run risks, alter the intertemporal distribution of income risk, by producing slow swings in the long-run growth of the endowments.

A growing body of the literature has focused on the relevance of low frequency risk in international finance. Bansal and Lundblad (2002) and Ammer and Mei (1996) pointed out that we need long-run highly correlated cash flows to reconcile the high degree of co-movement of international stock markets with the lack of correlation of fundamentals. Colacito and Croce (2007) and subsequently Colacito (2008) and Bansal and Shaliastovich (2006) have documented that the presence of slowly moving predictable components of consumption growths can help explaining the degree of volatility of exchange rates movements along with other major puzzles such as the Backus and Smith (1993) and the forward premium anomaly.[2] In all these papers, however, the optimal level of international trade is taken as given, meaning that all the asset pricing implications are derived working with exogenous post-trade aggregate consumption. Therefore these models cannot be used to gauge the welfare benefits of international risk-sharing. Our paper addresses this problem and shows that in order to correctly measure the benefits, it is important to account for long-run uncertainty.

The intuition behind our results is that financial integration leads international investors to benefit from increased risk-sharing opportunities at different frequencies. When a transitory shock hits one country, marginal utilities are almost unaffected, leaving little or no room to benefit from international risk-sharing. However when a long-run shock comes around and agents care about the temporal distribution of risk, the investors of that country experience a large jump in marginal utility: an important opportunity for international risk-sharing opens up, as long as the shock does not hit the two countries at the same time systematically.

[2]Alternatively, Verdelhan (2007) explains the uncovered interest rate puzzle in the contest of a habits model, while Fahri and Gabaix (2008) propose an explanation of international finance puzzles that relies on rare disasters.

The relevance of the long-run component is twofold. On the one hand it allows us to better quantify the total benefits of financial integration. On the other hand it allows us to produce reliable predictions on a wide set of observable variables. As argued by Baxter (1995), a challenge to existing theory is the volatility of net exports and exchange rates. In our model, the motive for risk-sharing induced by long-run risks, is also able to replicate the amount of volatility of net exports and terms of trade that we observe in the data for United States and United Kingdom. This is an important contribution of the paper, as we are able to account for all of these facts in the context of a frictionless complete markets model.

In addition, we show that the model is able to accurately replicate some of the stylized facts that characterize the transition from financial autarky to financial integration. Obstfeld (1998) suggests that capital mobility between US and UK accelerated starting in the late 1960s. If we calibrate the model to reflect the increased correlation of long-run risks after the Bretton-Woods era documented by Colacito and Croce (2007), we can account for the higher volatility of exchange rates fluctuations, the higher correlation and the lower volatility of consumption growths, during the years of financial integration.

In terms of economic theory, the use of non-time separable preferences poses a challenge in deriving the optimal allocations of the complete markets problem. We follow Kan (1995) and Anderson (2005) in recasting the problem as one in which the Pareto weights are time-varying. We show in the paper that the dynamics of the Pareto-weights is a leading force behind the ability of our model to generate results that are consistent with the data.

The paper is organized as follows. The next section describes the setup of the model and derives the equilibrium allocation both under the assumption of autarky and with complete markets. We then discuss the internal transmission mechanism of the model. Section 5 studies the welfare benefits of financial liberalization. We perform this exercise in two steps: first we assume that agents have risk-sensitive preferences, but endowments are only driven by transitory shocks, and then we introduce long-run risks. The following section documents the performance of the model in describing the behavior of international prices and quantities. The last section concludes the paper.

2 The economy

2.1 Preferences and endowments

There are two countries that we shall denote as home (h) and foreign (f) and two goods, whose endowment at each point in time will be denoted as X_t and Y_t respectively. Agents' preferences are defined over consumption aggregates of the two goods at each history. For exposition purposes, we shall focus on the following functional form.

Assumption 1 (Consumption aggregate)**.** *Let $\alpha \in (0,1)$. The consumption aggregate in the home and in the foreign countries are:*

$$C_t^h = \left[x_t^h\right]^{\alpha}\left[y_t^h\right]^{1-\alpha} \quad \textit{and} \quad C_t^f = \left[x_t^f\right]^{1-\alpha}\left[y_t^f\right]^{\alpha} \tag{1}$$

respectively.

Preferences are recursive, but non-time separable.

Assumption 2 (Preferences)**.** *Let $\gamma \geq 1$. Preferences in the home and in the foreign country have the following recursive representation:*

$$U_t^i = (1-\delta)\log C_t^i + \frac{\delta}{1-\gamma}\log E_t \exp\left\{(1-\gamma)\,U_{t+1}^i\right\} \tag{2}$$

$\forall i \in \{h, f\}$ *and for all histories.*

In what follows, we will denote $\theta = \frac{1}{1-\gamma}$. This is the same class of preferences studied among others by Anderson (2005) and Tallarini (2000) and correspond to a special case of Epstein and Zin (1989) in which the intertemporal elasticity of substitution parameter approaches 1.

Endowments follow an integrated process of order one. We also allow for the presence of explanatory variables.

Assumption 3 (Endowments' dynamics)**.** *Let the logarithm of X_t and Y_t follow the* *processes:*

$$
\begin{aligned}
\log X_t &= \mu_x + \log X_{t-1} + z_{1,t-1} + \varepsilon_{x,t} \\
\log Y_t &= \mu_y + \log Y_{t-1} + z_{2,t-1} + \varepsilon_{y,t} \\
z_{j,t} &= \rho_j z_{j,t-1} + \varepsilon_{j,t}, \forall j \in \{1,2\}
\end{aligned} \tag{3}
$$

and let $\Sigma = E[\xi'\xi]$*, with* $\xi = \begin{bmatrix} \varepsilon_1 & \varepsilon_2 & \varepsilon_x & \varepsilon_y \end{bmatrix}$.

This specification includes those used among others by Colacito and Croce (2007), Colacito (2008), and Bansal and Shaliastovich (2006).

2.2 The Pareto problem

We compute efficient allocations by solving a Pareto problem. For a given choice of weights $(\mu, 1-\mu)$, the planner's problem is:

$$
\begin{aligned}
\text{choose} \quad & \left\{x_t^h, x_t^f, y_t^h, y_t^f\right\}_{t=0}^{+\infty} \\
\text{to max} \quad & Q = \mu U_0^h + (1-\mu) U_0^f \\
\text{s.t.} \quad & x_t^h + x_t^f = X_t \\
& y_t^h + y_t^f = Y_t, \quad \forall t \geq 0
\end{aligned}
$$

In characterizing the equilibrium, we follow Anderson (2005) in formulating the problem in one in which the Pareto weights very over time. This takes into account the non-separability of the utility functions. We show in appendix that necessary conditions imply the following allocations:

$$
\begin{aligned}
x_t^h &= \frac{X_t}{1 + \frac{1-\alpha}{\alpha}\frac{\mu_t^f}{\mu_t^h}}, \qquad x_t^f = \frac{X_t}{1 + \frac{\alpha}{1-\alpha}\frac{\mu_t^h}{\mu_t^f}} \\
y_t^h &= \frac{Y_t}{1 + \frac{\alpha}{1-\alpha}\frac{\mu_t^f}{\mu_t^h}}, \qquad y_t^f = \frac{Y_t}{1 + \frac{1-\alpha}{\alpha}\frac{\mu_t^h}{\mu_t^f}}
\end{aligned} \tag{4}
$$

where

$$\mu_t^i = \mu_{t-1}^i \frac{\delta \exp\left\{\frac{U_t^i}{\theta}\right\} \pi_{t-1}}{E_{t-1} \exp\left\{\frac{U_t^i}{\theta}\right\}}, \quad \forall t \geq 1 \quad \text{and} \quad i \in \{h, f\} \tag{5}$$

and $\left(\mu_0^h, \mu_0^f\right) = (\mu, 1-\mu)$. In the appendix we show that the state variable $s_t = \log \mu_t^h / \mu_t^f$ can be approximated as

$$s_t = \rho_s s_{t-1} + M\varepsilon_t \tag{6}$$

where the autoregressive coefficient ρ_s is smaller than one.

We decentralize the economy by endowing the home country with good X and the foreign country with good Y, and by using one period state contingent Arrow-Debreu securities that can be traded in the world market and that are zero in net supply. The investor of the home country faces the price $Q_{t+1}^h(\zeta^{t+1})$ for the security that delivers one unit of good X_t on the occurrence of state ζ^{t+1}. The foreign consumer faces the price $Q_{t+1}^f(\zeta^{t+1})$ for the security that delivers one unit of good Y_t on the occurrence of the same state. As a consequence, the two budget constraint are

$$\begin{aligned} X_t^h + p_t Y_t^h + \int_{\zeta^{t+1}} A_{t+1}^h\left(\zeta^{t+1}\right) Q_{t+1}^h(\zeta^{t+1}) &= A_t^h + X_t \\ X_t^f + p_t Y_t^f + \int_{\zeta^{t+1}} A_{t+1}^f\left(\zeta^{t+1}\right) Q_{t+1}^f(\zeta^{t+1}) &= A_t^f + p_t Y_t \end{aligned} \tag{7}$$

where $A_t^i(\zeta^t)$ are holdings of the ζ^t contingent security by country i, and p_t is the relative price of goods Y and X. Market clearing implies that $A_t^h(\zeta^t) + A_t^f(\zeta^t) = 0$, $\forall d_t$.

2.3 Portfolio autarky

We follow Cole and Obstfeld (1991) in assuming that the home country's income is its endowment X_t, while the foreign country's income is Y_t. Trade is balanced in every period. Let p_t denote the price of good y in terms of good x. Then portfolio autarky features the following budget constraints

$$x_t^h + p_t y_t^h = X_t \tag{8}$$
$$x_t^f + p_t y_t^f = p_t Y_t \tag{9}$$

for the home and the foreign countries, respectively.

Definition 1 (Equilibrium with portfolio autarky)**.** *An equilibrium with portfolio autarky is a sequence of allocations* $\left\{x_t^h, x_t^f, y_t^h, y_t^f\right\}_{t=0}^{\infty}$ *and a price system* $\{p_t\}_{t=0}^{\infty}$ *such that, given prices, agents maximize (2) subject to budget constraints (8) and (9) and all markets clear.*

In appendix we show that the following is an equilibrium:

$$\begin{aligned} x_t^h = \alpha X_t, \qquad x_t^f = (1-\alpha) X_t \\ y_t^h = (1-\alpha) Y_t, \qquad y_t^f = \alpha Y_t \end{aligned} \tag{10}$$

and $p_t = X_t / Y_t$.

2.4 Solution of the model

Under the assumption of portfolio autarky, consumption growth evolves over time as a weighted average of the growths of the two endowments:

$$\begin{aligned} \Delta c_t^{h,aut} &= \alpha \Delta x_t + (1-\alpha)\Delta y_t \\ \Delta c_t^{f,aut} &= (1-\alpha)\Delta x_t + \alpha \Delta y_t \end{aligned}$$

In the appendix we show that with complete markets, there is an additional term entering the dynamics of consumption growth in addition to the one impelled by portfolio autarky:

$$\begin{aligned} \Delta c_t^h &= \Delta c_t^{h,aut} + \lambda_c^h s_{t-1} + \lambda_{cc}^h s_{t-1}^2 + \lambda_s^h (s_{t-1}) \varepsilon_t \\ \Delta c_t^f &= \Delta c_t^{f,aut} + \lambda_c^f s_{t-1} + \lambda_{cc}^f s_{t-1}^2 + \lambda_s^f (s_{t-1}) \varepsilon_t \end{aligned}$$

where s_t is defined as in (6). This changes the dynamics of consumption in at least three significant ways. First, it endogenously introduces an additional slowly moving predictive component of consumption growth, in addition to the two exogenous long-run risks. Second, it implies that consumption responds immediately to news about future long-run growth prospects, as opposed to the basic Bansal and Yaron (2004) model, in which there is only a lagged response. Third, it endogenously introduces a time-varying volatility term through the non-linear way in which s_t enters the optimal allocations.

In autarky, exchange rates are computed as the relative prices of the consumption bundles in the two countries. Given our choice of Cobb-Douglas aggregates, exchange rates are equal to the relative supply of the two goods in the world markets. Hence:

$$\Delta e_t \;=\; \Delta x_t - \Delta y_t \tag{11}$$

For the case of complete markets, the dynamics of exchange rates is derived by no arbitrage as the difference of the logarithm of the marginal rates of substitution m_t^h and m_t^f:

$$\Delta e_t \;=\; m_t^f - m_t^h \tag{12}$$

where

$$\begin{aligned} m_t^i &= \log \frac{\partial U_t^i / \partial C_t^i}{\partial U_t^i / \partial C_{t-1}^i} \\ &= \log \delta - \Delta c_t^i - \log \frac{\exp\{U_t^i/\theta\}}{E_{t-1}\exp\{U_t^i/\theta\}}, \quad \forall i \in \{h, f\} \end{aligned}$$

By definition, net exports are by definition equal to the value of exports minus the value of imports. With portfolio autarky and Cobb-Douglas consumption aggregates, the relative price of the two goods adjusts so that the net exports equal zero at each point in time. We show in the appendix that under the complete market assumptions the net export-output ratio for the home country is:

$$\frac{NX_t}{X_t} \;=\; \frac{1 - \mu_t^h/\mu_t^f}{1 - \frac{\alpha}{1-\alpha}\mu_t^h/\mu_t^f} \tag{13}$$

Hence a volatile time-varying ratio of Pareto weights introduce time variation in the net export-output ratio as well.

The price of the Arrow-Debreu security in the home country is

$$Q^h_{t+1}\left(\zeta^{t+1}\right) = \delta \exp\left\{-\Delta x^h_{t+1}\right\} \frac{\exp\left\{\frac{U^h_{t+1}+\Delta c^h_{t+1}}{\theta}\right\}}{E_t \exp\left\{\frac{U^h_{t+1}+\Delta c^h_{t+1}}{\theta}\right\}} \pi_{t+1|t}$$

and the price faced by the foreign country is pin down by no arbitrage:

$$Q^f_{t+1} = Q^h_{t+1} \frac{p_{t+1}}{p_t}$$

The international savings of the home country are directly related to the ratio of the Pareto weights:

$$\frac{A^h_t}{X_t} = \frac{\overline{W^h}}{X} \underbrace{f(s_t)}_{+}$$

where $\frac{\overline{W^h}}{X}$ is the unconditional mean of the wealth-output ratio in the home country and $f(s_t)$ is a function that we characterize in the appendix and that satisfies the following condition: $f(0) = 0$; $f(s)' > 0$.

3 Dynamics of the model

In this section we describe the dynamic response of the model to the various sources of shock. In figures 1 and 2 we show the response of domestic and foreign consumption growth and exchange rate depreciation rate both to a short-run (left panels) and a long-run (right panels) shock to home output. All variables are in log-units and are initialized at their own steady state values.

Under financial autarky there is no room for intertemporal consumption smoothing. By assumption, no asset can be traded across country and no debt can be issued. Trading in the goods market is limited because of the strong home-bias. Pareto weights are constant and the consumption growth rates implied by the optimal allocations in

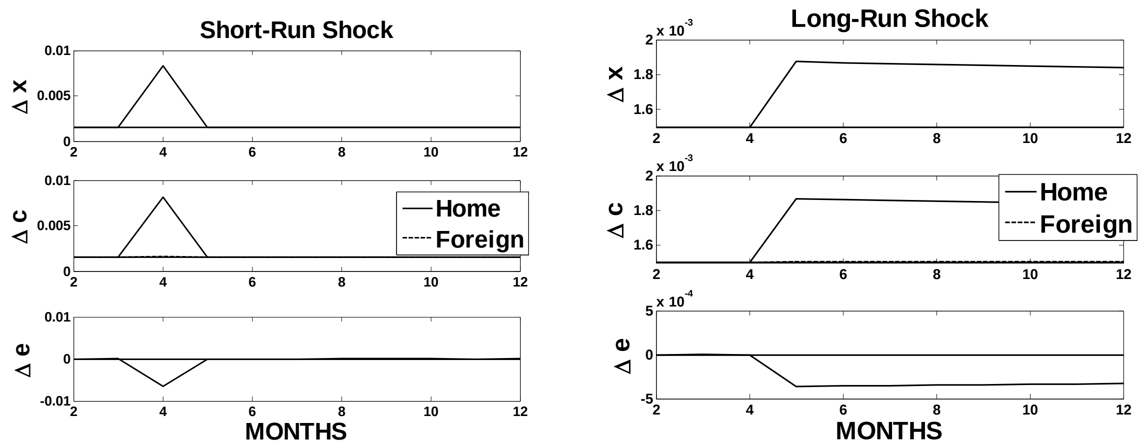

FIG. 1 - Fundamentals and exchange rate in Autarky. This figure shows impulse response functions for monthly output growth, Δx, consumption growth, Δc, and exchange rate depreciation, Δe. All the parameters are calibrated to the values reported in Table 1, pre-1970. The policy functions are computed assuming financial autarky. Shocks materialize at (and only at) time 4.

(10) are

$$\begin{aligned}\Delta c^h_{t+1} &= \alpha \Delta X_{t+1} + (1-\alpha) Y_{t+1}\\ \Delta c^f_{t+1} &= (1-\alpha)\Delta X_{t+1} + \alpha \Delta Y_{t+1},\end{aligned}$$

implying that home consumption growth has an almost one-to-one adjustment after domestic output shocks. Under financial autarky, exchange rate movements are completely driven by intratemporal trading in the goods market; because of the strong home-bias, the exchange rate adjustments are small.

The situation changes significantly when financial markets are complete. In this case, intertemporal consumption smoothing can be achieved by trading not only goods but also a state contingent international bond. In figure 3 we plot the response of the ratio of Pareto weights, the terms of trade and the net export-output ratio after both a short-run (left panels) and a long-run shock (right panels). Following a positive short-run shock to domestic income, the marginal utility of the domestic agent falls. The planner finds it optimal to reduce the relative weight on the home country and to give more resources to the foreign country, where the marginal utility has been almost unaffected. This adjustment produces an outflow of resources that translates in a positive adjustment in the net export-output ratio of the domestic country, and in an increase in foreign consumption. This explains why foreign consumption increases more than in autarky, while domestic consumption responds less. In the decentralized economy, the home country, indeed, finds it optimal to increase consumption less than proportionally and to lend resources to the foreign country. These savings will be used over time to sustain domestic growth.

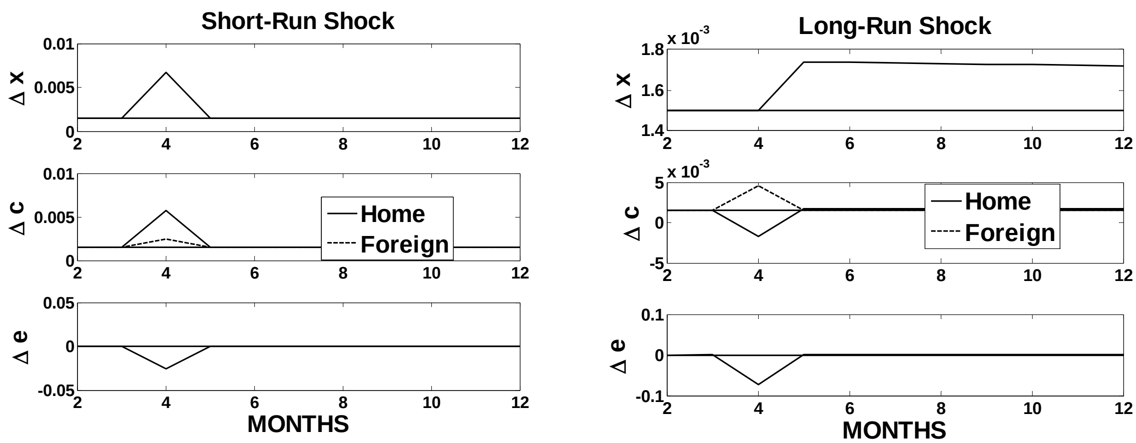

FIG. 2 - Fundamentals and exchange rate. This figure shows impulse response functions for monthly output growth, Δx, consumption growth, Δc, and exchange rate depreciation, Δe. All the parameters are calibrated to the values reported in Table 1, pre-1970. The policy functions are computed numerically. Shocks materialize at (and only at) time 4.

The planner's policy optimally smooths both home and foreign consumption growth over time. By reducing the current weight of the home country, and by promising a slowly increasing weight in the future, the planner reallocates consumption growth across countries and dates. Although a positive short-run shock increases home consumption by a smaller amount relative to autarky, its growth rate remains above the steady state for a very long time. This mechanism introduces a small endogenous and persistent moving average component in the dynamics of consumption growth, that resembles the exogenous one that is typically assumed in the long-run risk literature.

Following a long-run shock to the endowment, the consumption smoothing motives are even stronger for the country that receives the good news. The consumer antici-

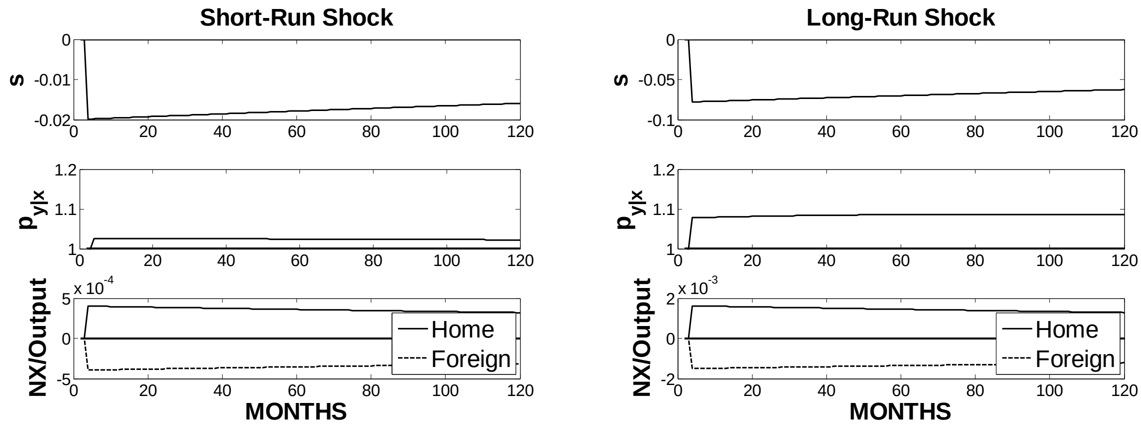

FIG. 3 - Net Export and Pareto Weights. This figure shows impulse response functions for the ratio of the pareto weights, s, the terms of trade, $p_{y|x}$, and the net-export-output ratio, $NX/Output$. All the parameters are calibrated to the values reported in Table 1, post-1970. The policy functions are computed numerically. Shocks materialize at (and only at) time 4.

pates the long-lasting increase in growth and experiences a significant improvement in her continuation value. Her current marginal utility drops and generates an incentive to consume less today (substitution effect).

The planner finds it optimal to immediately to reallocate resources toward the country that has not received the long-run shock and in which marginal utility is higher. The reallocation incentives are so strong that in the country affected by the positive long-run shock consumption drops more than proportionally. The savings accumulation process that is prompted by the decrease in consumption is used to sustain long-run consumption growth. Abroad, instead, consumption initially grows and then decreases over time as the planner starts increasing the Pareto weight of the home country. In the long-run less resources are being allocated to the foreign country and for this reason foreign consumption growth is negative while home consumption growth remains positive and higher than output growth.

In the decentralized economy, after good long-run news, the home country finds it optimal to increase its savings in order to sustain higher consumption growth in the long-run. The marginal utility of current consumption falls significantly and the agent decides to let her consumption drop.

When financial markets are open, exchange rate movements are significantly more sensitive to both short- and long-run endowment shocks. This can be explained by the fact that net export and terms of trade are more volatile (see figure 3).

4 The gains from risk sharing

In this section we investigate the welfare benefits that can be obtained by removing the portfolio autarky regime. We proceed in three steps. First we document the methodology that we adopt in order to compute the lifetime discounted utility associated with the allocations in the previous two sections. Then we quantify the welfare benefits under the assumption that the laws of motion of the endowments do not include long run risks. These risks are then added back in the concluding part of this section.

4.1 Methodology

We follow the literature on welfare costs and quantify the benefits of international diversification as the constant fraction of consumption that should be granted in every state and date of the world to make a representative consumer indifferent between having access or not to international financial markets. Let Δ be this percentage of consumption. Then

$$\Delta = \log \frac{C_t^{\Delta}}{C_t}$$

in every state and every date of the world. By solving the following two recursions:

$$\begin{aligned} U_t &= (1-\delta)\log C_t + \delta\theta \log E_t \exp\left\{\frac{U_{t+1}}{\theta}\right\} \\ U_t^{\Delta} &= (1-\delta)\left(\log C_t + \Delta\right) + \delta\theta \log E_t \exp\left\{\frac{U_{t+1}^{\Delta}}{\theta}\right\} \end{aligned}$$

the amount Δ can be computed as the difference between the two intercepts of the utility functions:

$$\Delta = A^{\Delta} - A \tag{14}$$

4.2 Cole and Obstfeld meet Tallarini

We start our analysis of the welfare gains of financial liberalizations by focusing on the special case in which the endowments follow pure random walk processes. This entails setting to zero the variances of the two shocks $\varepsilon_{1,t}$ and $\varepsilon_{2,t}$ in the system of equations (3) for all $t \geq 0$. This setup corresponds to a two country version of the model studied by Tallarini (2000). We report the results in figure 4. The benefits are plotted against two dimensions: the degree of home bias (α) and the coefficient of risk aversion (γ). In the figure, we let α range from 0.5 to 1.The results are symmetric when $\alpha \in [0, .5]$, given our modelling choice for the consumption aggregate. We set all the other parameters to the values reported in Table 1, pre-1970 calibration. We ultimately want to measure and characterize the potential benefits that the US and the UK had when they decided to open their financial markets.

Cole and Obstfeld (1991) pointed out that time-additive preferences would imply a negligible role for financial liberalization no matter the endowment process. Our results revisit their conclusion, by introducing a potentially important role for international financial markets. The benefits are largest for significant amounts of consumption home bias and are increasing in the coefficient of risk aversion. Both facts are intuitive. The closer is α to $1/2$, the more the model resembles a one good world. In this setup risks would be undiversifiable and hence nothing could be gained by allowing agents to trade financial assets across countries. On the other hand, when α is close to unity, agents display almost complete home bias and hence they would not benefit from exchanging claims on each other's endowments, because that would not have any significant impact on their consumption. Anywhere in between these two extrema, the benefits are positive. The higher γ, the more concerned agents are about the temporal distribution of risk. Hence there is room for international financial markets to improve on the welfare of the representative consumers of the two countries.

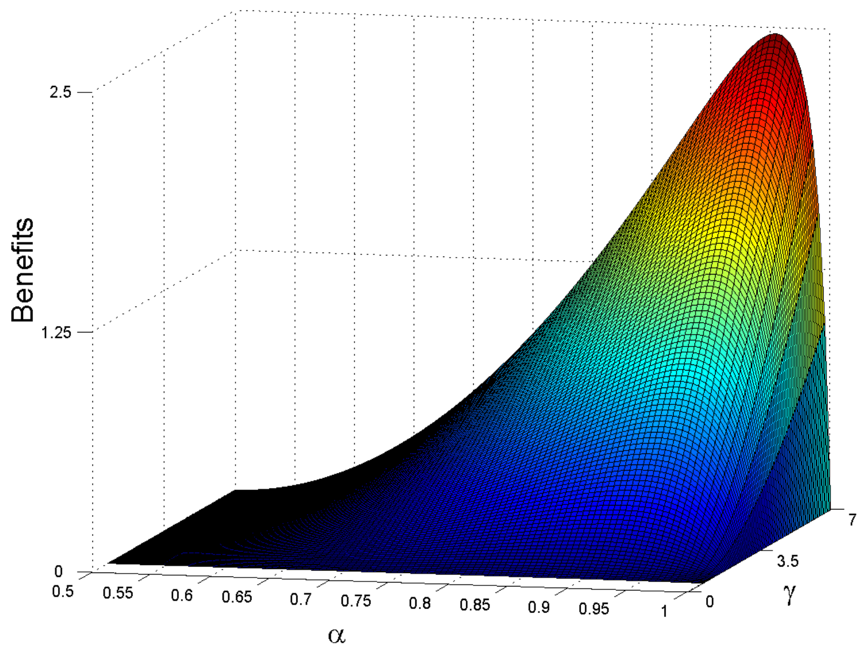

FIG. 4 - The benefits of international financial markets liberalization with no long run risks. The benefits are reported against increasing degrees of risk aversion (γ) and of home bias (α). All parameters are set to the values in Table 1, calibration pre-1970.

It is interesting to notice that the highest benefits are obtained for remarkably high degrees of consumption home bias. The evidence provided, among others, by Lewis (1999) suggests that this is the most empirically relevant case. It should also be noticed that moderate levels of risk aversion give rise to non negligible welfare benefits in the order of 2% of lifetime consumption. Hence our explanation is not entirely driven by implausible levels of risk aversion.

4.3 Cole and Obstfeld meet Bansal and Yaron

If we add slowly moving predictable components to the dynamics of the growth rates of the endowments, the welfare benefits increase dramatically, as suggested by figure 5. A ban on international trade of securities could result in a loss of up to 10% of lifetime consumption, a number significantly higher than what previously reported in

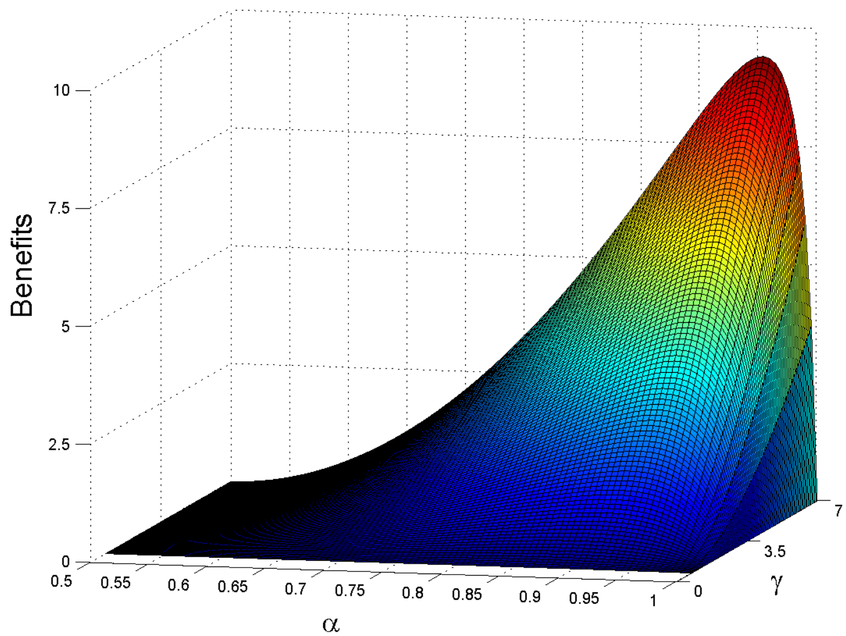

FIG. 5 - The benefits of international financial markets liberalization with long run risks. The benefits are reported against increasing degrees of risk aversion (γ) and of home bias (α). All the other parameters are set to the values reported in Table 1, calibration pre-1970.

the literature. Interestingly, the highest benefits lie in the region in which α is close to one, which is probably the most empirically relevant. The large increase in benefits after the introduction of long-run risks comes from the agents having a strong preference for the timing of the resolution of the uncertainty. Indeed if $\gamma \to 1$, the CRRA case attains and the benefits are close to zero. It is important that the sources of long-run growth fluctuations are not perfectly correlated across countries, as this would be an uninsurable source of risk and, as such, no further gain would come from international diversification. Nevertheless, we need these shocks to be highly correlated across countries to account for the relative smoothness of exchange rates movements, as documented in the next section.

5 Reconciling international quantities and prices

5.1 Calibration

Table 1 reports our calibrations. We labeled them as pre- and post-1970, because we follow Colacito and Croce (2007) in setting ρ_{12} to a higher number for the sample starting in 1970. This reflects the increased correlation of long-run growth perspective in the post–Bretton-Woods era. Correspondingly, we reduce ρ_{xy} in order to keep the international correlation of output growths constant. The coefficient of risk aversion γ is set to 7, which is on the down side of the literature on the equity premium puzzle. The monthly subjective discount factor is equal to 0.998, about .98 in annual terms. The parameters governing the dynamics of output growth are chosen to replicate as close as possible the first two moments and the autocorrelation functions that are observed in the data. A comparison of table 1 and table 2 shows that it is indeed the case that a small degree of autocorrelation cannot be rejected for the US and the UK.

5.2 Discussion

In table 1 we report the moments produced by simulating the model under both complete markets and financial autarky. We also report the equivalent moments observed in the data for US and UK in Table 2. The dataset is the same used by Colacito and Croce (2007), data are annual, and the sample is from 1930 to 2003. We think it is important to look also at two separate sub-samples: the post-Bretton Woods era (1971-2003), and the years 1948-1970. The choice of the two sub-samples is motivated by Obstfeld (1998), that suggests that there has been an increasing financial integration between US and UK, starting in the late 1960s.

TABLE 1
MODEL PREDICTIONS

Panel A

Calibration	μ	σ	σ_x	ρ	ρ_{12}	ρ_{xy}	α	δ^{12}	RRA	IES
Post-1970	.165%	.54%	4%σ	.988	.90	.05	.98	.988	7	1
Pre-1970	.165%	.54%	4%σ	.988	.50	-.50	.98	.988	7	1

Panel B

	Post-1970		Pre-1970	
	No Autarky	Autarky	No Autarky	Autarky
$Std[\Delta y^h]$	1.92	1.91	1.92	1.92
$ACF_1[\Delta y^h]$	.431	.431	0.43	0.43
$Std[\Delta c^h]$	1.78	1.88	2.98	1.88
$ACF_1[\Delta c^h]$	.463	.436	0.33	0.43
$Std[NX^h/Y^h]$	.81	0	3.60	0
$ACF_1[NX^h/Y^h]$	.847	-	0.80	-
$corr[\Delta y_t^h, \Delta c_t^h]$	.940	.991	0.53	0.99
$corr[\Delta y_t^h, \Delta y_t^f]$	.367	.360	-0.11	-0.11
$corr[\Delta c_t^h, \Delta c_t^f]$	.639	.392	0.03	-0.07
$E[r_f^h]$	2.70	2.72	2.05	2.71
$Std[r_f^h]$	1.21	1.18	2.25	1.21
$corr[r_{f,t}^h, r_{f,t}^f]$	0.87	0.89	-0.43	0.50
$Std[\Delta e]$	14.5	2.48	23.2	3.28
$Std[m^h]$	36.2	33.5	32.7	38.1
Total Benefits	4.39		8.62	
SRR Benefits	2.26		1.91	

Panel A shows the benchmark monthly calibrations we employ for the pre-Bretton-Woods sample (Pre-BW; 1945-1970) and the post-Bretton-Wood sample (Post-BW; 1971-2003). In panel B, all the statistics are annual and multiplied by 100 (except for correlations and auto-correlations). The entries are based on 1000 simulations each with 360 monthly observations that are time-aggregated to an annual frequency.

Output is measured as the sum of local consumption and local net export. We eliminate investment and government expenditure in order to isolate the role of international trade on consumption smoothing. We employ total net exports for both US and

UK.

TABLE 2
US AND UK DATA

	1946-2003		1946-1970		1971-2003	
	US	UK	US	UK	US	UK
$Std[\Delta y^h]$	1.91	4.33	2.57	5.98	1.17	2.56
$ACF_1[\Delta y^h]$	.067	-.047	.087	-.068	.003	.042
$Std[\Delta c^h]$	1.30	2.78	1.49	3.35	1.14	2.27
$ACF_1[\Delta c^h]$	.179	.320	.057	.205	.333	.459
$Std[NX^h/Y^h]$	2.85	3.05	1.62	2.61	2.61	3.43
$ACF_1[NX^h/Y^h]$	.840	.617	.658	.370	.805	.695
$corr[\Delta y_t^h, \Delta c_t^h]$	.729	.753	.901	.902	.635	.510
$corr[\Delta y_t^h, \Delta y_t^f]$	.485		.529		.317	
$corr[\Delta c_t^h, \Delta c_t^f]$	.435		.361		.584	
$E[r_f^h]$	.694	1.23	.911	1.10	1.07	1.52
$Std[r_f^h]$	1.58	1.12	1.90	.858	1.18	1.22
$corr[r_{f,t}^h, r_{f,t}^f]$	.498		.279		.672	
$Std[\Delta e]$	10.42		8.12		11.90	

All the statistics are annual and multiplied by 100 (except for correlations and autocorrelations). Data are annual, more details can be found in the Appendix.

There are several empirical facts that the model is able to reproduce. Contrary to the standard time-additive log-preferences case, when markets are complete we are able to get a volatile and very persistent net export-output ratio without needing any real or nominal friction. We regard this as one of the main contributions of the paper, as the literature so far has struggled in producing any net-export dynamics at all.

The data suggests that financial integration should reduce the contemporaneous correlation between domestic output and domestic consumption, while at the same time increasing the correlation of consumption growth rates across countries. Our model appears to do a remarkably good job along all these dimensions.

As far as prices are concerned, the model in which markets are complete delivers an exchange rate volatility that is in line with the data, while this series is very smooth in the model with financial autarky. This is an interesting finding, as we can think of the former as the relatively high volatility during the post–Bretton-Woods era and the latter as the low volatility that characterized the fixed nominal exchange regime period.

The mean of the risk free rates is about 1% higher than that observed in the data. In the long-run risk literature this problem is solved by calibrating the intertemporal elasticity of substitution to a number above one in order to reduce the impact of growth on the level of the risk-free rate. We regard as an interesting generalization of this analysis, the case of intertemporal elasticity of substitution different from one. The model implied volatility of the risk free rate is low and such consistent with what suggested by the data. The risk free rates in the two countries have a cross-country correlation slightly higher than what observed in the data. Colacito and Croce (2007) argue that this due to the use of a symmetric calibration and a more general parameterization that takes into account country specific difference would deliver the right amount of correlation of real risk-free rates.

In addition the model is able to produce a volatility of the stochastic discount factors that is at least as high as impelled by the Hansen and Jagannathan (1991) bound. This result generalizes the finding of Bansal and Yaron (2004) to the cross-section of countries, and was pointed out also by Colacito and Croce (2007) and Colacito (2008). The interesting addition to the literature is that we are endogenously able to generate stochastic discount factors that are this volatile.

6 Concluding remarks

In this paper we have developed a general equilibrium model that is able to account for a number of quantitatively challenging facts of international finance. More precisely, the model generated series agree with both international prices and quantities, a finding new to the international finance literature. We take this model as one in which we can address the question of how important financial integration is. It turns out that these benefits can be as large as 10% of lifetime consumption, once the intertemporal distribution of output risk is taken into account. In particular, we show that an overwhelmingly high share of the benefits has to do with risk-sharing for the long-run.

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Appendix

Alternative solution algorithm

In this section we develop general results obtained by log-linearizing an asset pricing model with Epstein-Zin preferences and a linear state space representation for consumption and dividends growth.

Endowment and Consumption Flows

Assume the following linear state space for the growth rates of the two goods in the economy:

$$\begin{aligned}
\Delta x_t &= \mu_x + z_{1,t-1} + \varepsilon_{x,t} \\
\Delta y_t &= \mu_y + z_{2,t-1} + \varepsilon_{y,t} \\
z_{j,t} &= \rho_j z_{j,t-1} + \varepsilon_{j,t}, \forall j \in \{1,2\}
\end{aligned} \tag{15}$$

with

$$\varepsilon_t \equiv \begin{bmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \\ \varepsilon_{x,t} \\ \varepsilon_{y,t} \end{bmatrix} \sim iidN(0_4, \Sigma)$$

Define $S_t \equiv \frac{\mu_t^h}{\mu_t^f}$ and $\kappa \equiv \alpha/(1-\alpha)$. For given S_t, the optimal allocation (capital letters indicate variables in raw units) implies:

$$
\begin{aligned}
X_t^h &= \frac{\kappa S_t}{1+\kappa S_t} X_t, \qquad X_t^f = \frac{1}{1+\kappa S_t} X_t \\
Y_t^h &= \frac{S_t/\kappa}{1+S_t/\kappa} Y_t, \qquad Y_t^f = \frac{1}{1+S_t/\kappa} Y_t
\end{aligned} \tag{16}
$$

Assume the unconditional mean of S_t exists and denote it as $\overline{S}$. Define the following parameters:

$$
\begin{aligned}
\lambda_x^h &= \frac{1}{1+\kappa\overline{S}} \qquad & \lambda_{xx}^h &= -\frac{\kappa\overline{S}}{(1+\kappa\overline{S})^2} \\
\lambda_x^f &= -\frac{\kappa\overline{S}}{1+\kappa\overline{S}} \qquad & \lambda_{xx}^h &= -\frac{\kappa\overline{S}}{(1+\kappa\overline{S})^2} \\
\lambda_y^h &= \frac{1}{1+\overline{S}/\kappa} \qquad & \lambda_{yy}^f &= -\frac{\overline{S}/\kappa}{(1+\overline{S}/\kappa)^2} \\
\lambda_y^f &= -\frac{\overline{S}/\kappa}{1+\overline{S}/\kappa} \qquad & \lambda_{yy}^h &= -\frac{\overline{S}/\kappa}{(1+\overline{S}/\kappa)^2}
\end{aligned} \tag{17}
$$

Define $s_t \equiv \log(S_t) - \log(\overline{S})$. A second order log-expansion of the allocation around the unconditional mean of S_t implies that:

$$
\begin{aligned}
\Delta x_{t+1}^h &= \Delta x_{t+1} + \lambda_x^h \Delta s_{t+1} + .5\lambda_{xx}^h (\Delta s_{t+1})^2 \\
\Delta y_{t+1}^h &= \Delta y_{t+1} + \lambda_y^h \Delta s_{t+1} + .5\lambda_{yy}^h (\Delta s_{t+1})^2 \\
\Delta x_{t+1}^f &= \Delta x_{t+1} + \lambda_x^f \Delta s_{t+1} + .5\lambda_{xx}^f (\Delta s_{t+1})^2 \\
\Delta y_{t+1}^f &= \Delta y_{t+1} + \lambda_y^f \Delta s_{t+1} + .5\lambda_{yy}^f (\Delta s_{t+1})^2
\end{aligned}
$$

From now on, assume the following growth rate for s_t:

$$\Delta s_{t+1} \approx (\rho_s - 1)s_t + M\varepsilon_{t+1} \tag{18}$$

where M is a 1x4 vector that I will derive later.

Consumption in the two countries is defined as:

$$\begin{aligned}
C_t^i &= [X_t^i]^{\alpha_i}[Y_t^i]^{1-\alpha_i}, \forall i \in \{h, f\} \\
\alpha_h &= \alpha \\
\alpha_f &= 1 - \alpha
\end{aligned}$$

In order to simplify the notation, define the following parameters:

$$\begin{aligned}
\lambda_c^i &= (\alpha_i\lambda_x + (1-\alpha_i)\lambda_y)(\rho_s - 1),\ \forall i \in \{h, f\} \\
\lambda_{cc}^i &= (\rho_s^2 - 1)(\alpha_i\lambda_{xx} + (1-\alpha_i)\lambda_{yy}) \\
\lambda_s^i(s_t) &= \alpha_i\lambda_x + (1-\alpha_i)\lambda_y + (\alpha_i\lambda_{xx} + (1-\alpha_i)\lambda_{yy})\rho_s s_t
\end{aligned} \tag{19}$$

At this point we can write:

$$\Delta c_{t+1}^i \approx \alpha_i\Delta x_{t+1} + (1-\alpha_i)\Delta y_{t+1} + \lambda_c^i s_t + .5\lambda_{cc}^i s_t^2 + \lambda_s^i(s_t)M\varepsilon_{t+1}$$

Value Function

The recursion for the log of the utility-consumption ratio is:

$$u_t^i = \frac{\delta^i}{1-\gamma} \log\left(E_t[\exp^{(1-\gamma)(u_{t+1}^i + \Delta c_{t+1}^i)}]\right) \qquad \forall i \in \{h, f\}$$

Let's guess that:

$$u_t^i \;=\; A^i + B_0^i s_t + B_1^i z_{1,t} + B_2^i z_{2,t} + B_3^i s_t^2 \quad \forall i \in \{h, f\}$$

In order to evaluate the right-hand side of the recursion it is convenient to impose $u_{t+1}^i + \Delta c_{t+1}^i \equiv E_t^i + R_{t+1}^i \ \forall i \in \{h, f\}$. E_t^i contains all the terms that can be predicted at time t and R_{t+1}^i contains all the terms involving future news. For country $i \in \{h, f\}$ we have:

$$\begin{aligned} E_t^i &= A^i + B_0^i \rho_s s_t + B_1^i \rho_1 z_{1,t} + B_2^i \rho_2 z_{2,t} + B_3^i \rho_s^2 s_t^2 + \lambda_c^i s_t + .5\lambda_{cc}^i s_t^2 \\ &+ (\alpha_i \mu_x + (1-\alpha_i)\mu_y) + \alpha_i z_{1,t} + (1-\alpha_i) z_{2,t} \end{aligned}$$

$$\begin{aligned} R_{t+1}^i &= \Gamma_{r,\varepsilon}^i(s_t)\varepsilon_{t+1} + B_3^i (M\varepsilon_{t+1})^2 \\ \Gamma_{r,\varepsilon}^i(s_t) &\equiv [B_0^i + \lambda_s(s_t) + 2B_3\rho_s s_t]M + \begin{bmatrix} B_1^i & B_2^i & \alpha_i & 1-\alpha_i \end{bmatrix} \end{aligned}$$

According to Jacobson(1978), the recursion can be written as:

$$u_t^i \approx \delta^i \left(E_t^i + .5\left[(1-\gamma)\Gamma_{r,\varepsilon}^i(s_t)\Sigma\Gamma_{r,\varepsilon}^i(s_t)' - \frac{1}{1-\gamma}\log[det(I - 2(1-\gamma)^2 (B_3^i)^2 M\Sigma M')] \right] \right)$$

Notice that $\Gamma_{r,\varepsilon}^i(s_t)\Sigma\Gamma_{r,\varepsilon}^i(s_t)'$ is a quadratic form in s_t and for this reason it is important to compute it explicitly before matching the value function coefficients.

According to our assumptions:

$$\Sigma = \begin{bmatrix} \Sigma_{12} & 0_2 \\ 0_2 & \Sigma_{34} \end{bmatrix}, \quad \Sigma_{12} = \begin{bmatrix} \sigma_1^2 & \sigma_1\sigma_2\rho_{12} \\ \sigma_1\sigma_2\rho_{12} & \sigma_2^2 \end{bmatrix}, \quad \Sigma_{xy} = \begin{bmatrix} \sigma_x^2 & \sigma_x\sigma_y\rho_{xy} \\ \sigma_x\sigma_y\rho_{xy} & \sigma_y^2 \end{bmatrix}$$

Now, let's rewrite $\Gamma^i_{r,\varepsilon}(s_t)$ in a more convenient way:

$$\Gamma^i_{r,\varepsilon}(s_t)' \equiv \begin{bmatrix} \phi^i_1 + \omega^i_1 s_t \\ \phi^i_2 + \omega^i_2 s_t \\ \phi^i_3 + \omega^i_3 s_t \\ \phi^i_4 + \omega^i_4 s_t \end{bmatrix}$$

where

$$\begin{aligned} \omega^i_j &= \rho_s \left[(\alpha_i\lambda^i_{xx} + (1-\alpha_i)\lambda^i_{yy})M_j + 2B^i_3\right], \ j = 1, ..., 4 \qquad (20) \\ \phi^i_j &= B_0 M_j + B^i_j + M_j(\alpha_i\lambda^i_x + (1-\alpha_i)\lambda^i_y), \ j = 1, 2 \\ \phi^i_3 &= B_0 M_3 + \alpha_i + M_3(\alpha_i\lambda^i_x + (1-\alpha_i)\lambda^i_y) \\ \phi^i_4 &= B_0 M_4 + 1 - \alpha_i + M_4(\alpha_i\lambda^i_x + (1-\alpha_i)\lambda^i_y) \end{aligned}$$

In order to keep the notation compact, define the following scalars:

$$\begin{aligned} \Phi^i &= [\phi^i_1 \ \phi^i_2]\Sigma_{12}[\phi^i_1 \ \phi^i_2]' + [\phi^i_3 \ \phi^i_4]\Sigma_{xy}[\phi^i_3 \ \phi^i_4]' \qquad (21) \\ \Theta^i &= [\omega^i_1 \ \omega^i_2]\Sigma_{12}[\omega^i_1 \ \omega^i_2]' + [\omega^i_3 \ \omega^i_4]\Sigma_{xy}[\omega^i_3 \ \omega^i_4]' \\ \Omega^i &= [\phi^i_1 \ \phi^i_2]\Sigma_{12}[\omega^i_1 \ \omega^i_2]' + [\phi^i_3 \ \phi^i_4]\Sigma_{xy}[\omega^i_3 \ \omega^i_4]' \end{aligned}$$

This implies:

$$\Gamma^i_{r,\varepsilon}(s_t)\Sigma\Gamma^i_{r,\varepsilon}(s_t)' = \Phi^i + \Omega^i s_t + \Theta^i s_t^2$$

At this point, the coefficients of the value functions can be characterized by the following conditions:

$$\begin{aligned}
A^i &= \delta^i(A^i + \alpha_i\mu_x + (1-\alpha_i)\mu_y + .5(1-\gamma)\Phi^i + ... \\
&- .5\frac{1}{1-\gamma}\log[\det(I - 2(1-\gamma)^2(B_3^i)^2 M\Sigma M')]) \\
B_0^i &= \delta^i\left(\rho_s B_0^i + \lambda_c^i + .5(1-\gamma)\Omega^i\right) \\
B_1^i &= \delta^i\left(\rho_1 B_1^i + \alpha^i\right) \\
B_2^i &= \delta^i\left(\rho_1 B_2^i + 1 - \alpha^i\right) \\
B_3^i &= \delta^i\left(\rho_s^2 B_3^i + .5(1-\gamma)\Theta^i\right),\ \forall i \in \{h, f\}
\end{aligned} \tag{22}$$

The Evolution of s_t

As derived before:

$$S_{t+1} = S_t \frac{\delta^h \frac{\exp\left\{\frac{U^h_{t+1}}{\theta}\right\}}{E_t \exp\left\{\frac{U^h_{t+1}}{\theta}\right\}}}{\delta^f \frac{\exp\left\{\frac{U^f_{t+1}}{\theta}\right\}}{E_t \exp\left\{\frac{U^f_{t+1}}{\theta}\right\}}} \tag{23}$$

According to our approximation for the value function:[3]

$$\Delta s_{t+1} \approx (1-\gamma)(R^h_{t+1} - R^f_{t+1}) - .5(1-\gamma)^2 \left\{ \Gamma^h_{r,\varepsilon} \Sigma \Gamma^{h'}_{r,\varepsilon} - \Gamma^f_{r,\varepsilon} \Sigma \Gamma^{f'}_{r,\varepsilon} \right\} \tag{24}$$

Matching the previous equation with our guess for Δs_{t+1} implies:

$$\begin{aligned} \rho_s &\approx 1 - .5(1-\gamma)^2[\Omega^h - \Omega^f] \\ M' &= \begin{bmatrix} \frac{B^h_1 - B^f_1}{(1-\gamma)^{-1} - [B^h_0 - B^f_0 - (\lambda^h_s(0) - \lambda^f_s(0))]} \\ \frac{B^h_2 - B^f_2}{(1-\gamma)^{-1} - [B^h_0 - B^f_0 - (\lambda^h_s(0) - \lambda^f_s(0))]} \\ \frac{\alpha_h - \alpha_f}{(1-\gamma)^{-1} - [B^h_0 - B^f_0 - (\lambda^h_s(0) - \lambda^f_s(0))]} \\ \frac{\alpha_f - \alpha_h}{(1-\gamma)^{-1} - [B^h_0 - B^f_0 - (\lambda^h_s(0) - \lambda^f_s(0))]} \end{bmatrix} \end{aligned} \tag{25}$$

Solution Method

All the coefficients can be found by recursively iterating on (23). In particular, guess B^i_j $\forall i \in \{h, f\}$ and $j \in \{1, ..., 4\}$ and $\rho_s : 0 \leq \rho_s < 1$, compute all the coefficients and update the guess according to (17)-(25).

Decentralized Economy

Consumption Pricing Kernel

[3]Notice that s_{t+1} was defined in terms of log deviations of S_t from its unconditional mean and for this reason it is possible to neglect the constant terms.

The stochastic discount factor with respect to the composite consumption good is:

$$
\begin{aligned}
m^i_{t+1} &= -\Delta c^i_{t+1} + \Delta \log \mu^i_{t+1} \\
&\approx -\{\alpha_i \Delta x_{t+1} + (1-\alpha_i)\Delta y_{t+1} + \lambda^i_c s_t + .5\lambda^i_{cc} s_t^2 + \lambda^i_s M \varepsilon_{t+1}\} + ... \\
... &\quad + \log(\delta^i) + (1-\gamma)\left(\Gamma^i_{r,\varepsilon}\varepsilon_{t+1} + B^i_3 (M\varepsilon_{t+1})^2\right) + ... \\
... &\quad -.5(1-\gamma)^2(\Phi^i + \Omega^i s_t + \Theta^i s_t^2) + ... \\
... &\quad +.5\log[\det(I - 2(1-\gamma)^2 (B^i_3)^2 M\Sigma M')]
\end{aligned}
$$

In a more compact form:

$$
\begin{aligned}
m^i_{t+1} &\approx \overline{m}^i + \Gamma^i_{m,s}[z_{1,t} \quad z_{2,t} \quad s_t \quad s_t^2]' + \Gamma^i_{m,\varepsilon}(s_t)\varepsilon_{t+1} + (1-\gamma)B^i_3(M\epsilon_{t+1})^2 \\
&\text{where} \\
\Gamma^{i'}_{m,s} &\equiv \begin{bmatrix} -\alpha_i \\ -(1-\alpha_i) \\ -\lambda^i_c - .5(1-\gamma)^2\Omega^i \\ -.5(\lambda^i_{cc} + (1-\gamma)^2\Theta^i) \end{bmatrix} \\
\Gamma^i_{m,\varepsilon}(s_t) &\equiv \lambda^i_s(s_t)M + (1-\gamma)\Gamma^i_{r,\varepsilon}(s_t) + [0 \quad 0 \quad -\alpha_i \quad -(1-\alpha_i)] \\
\overline{m} &= \log(\delta^i) - (\alpha^i\mu_x + (1-\alpha^i)\mu_y) - .5(1-\gamma)^2\Phi^i + ... \\
... &\quad +.5\log[\det(I - 2(1-\gamma)^2(B^i_3)^2 M\Sigma M')]
\end{aligned}
$$

Risk Free Rate

A Bond that pays a unit of composite consumption after one period will have the following risk free rate:

$$r_t^{f,i} \equiv -\log E_t\left[\exp^{m_{t+1}^i}\right]$$

Using our model:

$$\begin{aligned}
r_t^{f,i} &\approx \overline{r}_t^{f,i} + \Gamma_{c,s}^i [z_{1,t} \quad z_{2,t} \quad s_t \quad s_t^2]' \\
\Gamma_{c,s}^{i'} &= [\alpha_i \quad 1-\alpha_i \quad \lambda_c^i \quad \lambda_{cc}^i] \\
\overline{r}_t^{f,i} &= -\overline{m}^i - .5\Gamma_{m,\varepsilon}^i(s_t)\Sigma\Gamma_{m,\varepsilon}^{i'}(s_t) + .5\log[\det(I - 2(1-\gamma)^2(B_3^i)^2 M\Sigma M')]
\end{aligned}$$

Real Exchange Rate

The real exchange rate is defined as the relative price of domestic versus foreign consumption:

$$E_t \equiv \frac{p_{c,t}^h}{p_{c,t}^f}$$

where $p_{c,t}^i$ is the price of a unit of consumption in country $i \in \{h, f\}$ and is computed as:

$$p_{c,t}^i \equiv \frac{X_t^i + p_{y,t} Y_t^i}{C_t^i} \quad i \in \{h, f\}$$

We let $p_{y,t}$ denote the terms of trade between foreign and domestic goods. We determine $p_{y,t}$ in the next section.

If markets are complete, the depreciation of the real exchange rate can also be written as follows:

$$
\begin{aligned}
\Delta e_{t+1} &= m_{t+1}^{h} - m_{t+1}^{f} \\
&\approx \overline{m}^{h} - \overline{m}^{f} + \Gamma_{e,s}[z_{1,t} \quad z_{2,t} \quad s_t \quad s_t^2]' + \Gamma_{e,\varepsilon}\epsilon_{t+1}
\end{aligned}
$$

where

$$
\begin{aligned}
\Gamma_{e,s} &\equiv \Gamma_{m,s}^{h} - \Gamma_{m,s}^{f} \\
\Gamma_{e,\varepsilon} &\equiv \Gamma_{m,\varepsilon}^{h} - \Gamma_{m,\varepsilon}^{f}
\end{aligned}
$$

Net Exports-Output Ratio and terms of trade

According to our model, the Net Exports for the home country are:

$$
NX_t^h = X_t^f - p_{y,t}Y_t^h = \left(\frac{1}{\kappa S_t} - \frac{1}{\kappa}\right)\frac{1}{1+\frac{1}{\kappa S_t}}X_t
$$

This implies that:

$$
\begin{aligned}
\frac{NX_t^h}{X_t} &= \frac{1-S_t}{1+\kappa S_t} \\
\frac{d\frac{NX^h}{X}}{dS} &= -\frac{1+\kappa}{(1+\kappa S)^2} < 0
\end{aligned}
$$

The terms of trade can be recovered from the marginal rate of substitution of the agent in the home country:

$$
\begin{aligned}
p_{y,t} &= \frac{1-\alpha}{\alpha}\frac{X_t^h}{Y_t^h} \\
&= \frac{1-\alpha}{\alpha}\frac{X_t}{Y_t}\frac{\kappa+\kappa S_t}{1+\kappa S_t}
\end{aligned}
$$

Debt-Output Ratio

The budget constraint for the home country is:

$$
X_t^h + p_{y,t} Y_t^h + \int_{\zeta^{t+1}} A_{t+1}^h(\zeta^{t+1}) Q_{t+1}^h(\zeta^{t+1}) = A_t^h + X_t
$$

where A_t^h is the level of international savings of the home country at time t after history ζ^t; $A_{t+1}^h(\zeta^{t+1})$ is the number securities paying a unitary payoff at time $t+1$ if history $\zeta^{t+1} = \zeta^t | \zeta_{t+1}$ is realized; and $Q_{t+1}^h(\zeta^{t+1})$ is the time t domestic price of the correspondent security.

In a similar way, the budget constraint of the foreign country is:

$$
X_t^f + p_{y,t} Y_t^f + \int A_{t+1}^f Q_{t+1}^f = A_t^f + Y_t
$$

The market clearing condition for the state-contingent securities is:

$$
A_t^h(\zeta^t) = -A_t^f(\zeta^t) \quad \forall \zeta^t
$$

Using the FOCs of the representative agent in the home country:

$$
Q_{t+1}^h = \delta^h \exp^{-\Delta x_{t+1}^h} \frac{\exp^{(1-\gamma)(u_{t+1}^h + \Delta c_{t+1}^h)}}{E_t[\exp^{(1-\gamma)(u_{t+1}^h + \Delta c_{t+1}^h)}]} \pi_{t+1|t}
$$

By no arbitrage:

$$
\begin{aligned}
Q^f_{t+1} &= Q^h_{t+1} \frac{p_{y,t+1}}{p_{y,t}} \\
&= \delta^f \exp^{-\Delta y^f_{t+1}} \frac{\exp^{(1-\gamma)(u^f_{t+1}+\Delta c^f_{t+1})}}{E_t[\exp^{(1-\gamma)(u^f_{t+1}+\Delta c^f_{t+1})}]} \pi_{t+1|t}
\end{aligned}
$$

Notice that all the variables are expressed in terms of home-good units. In the model there is growth, but the debt-output ratio, $a^h_t \equiv \frac{A^h_t}{X_t}$, is stationary:

$$
(1+\kappa)\frac{S_t}{1+\kappa S_t} + \int a^h_{t+1} Q^h_{t+1} \exp^{\Delta x_{t+1}} = a^h_t + 1
$$

Solving forward the budget constraint implies:

$$
a^h_t = \mathbf{EPV}_t \left((1+\kappa)\frac{S_t}{1+\kappa S_t} \right) - \mathbf{EPV}_t \left(1\right)
$$

where EPV stands for Expected Present Value. In order to compute a^h_t in closed form, it is necessary to compute the two expected present values above. We write them in a recursive form and we log linearize them. Before continuing, it is convenient to define the following growth-adjusted discount factor:

$$
\begin{aligned}
m_{x,t+1} &\equiv q^h_{t+1} + \Delta x_{t+1} = \Delta\mu^h_{t+1} - \Delta x^h_{t+1} + \Delta x_{t+1} \\
&\approx \Delta\mu^h_{t+1} - (\lambda^h_x \Delta s_{t+1} + .5\lambda^h_{xx}\Delta s^2_{t+1}) \\
&\approx \overline{m}_x + \Gamma_{m_x,s}[z_{1,t} \quad z_{2,t} \quad s_t \quad s^2_t]' + \Gamma_{m_x,\varepsilon}\varepsilon_{t+1} + \Gamma_{m_x,M\varepsilon}(M\varepsilon_{t+1})^2
\end{aligned}
$$

where

$$
\begin{aligned}
\Gamma'_{m_x,s} &= \begin{bmatrix} 0 \\ 0 \\ -\lambda^h_x(\rho_s - 1) - .5(1-\gamma)^2\Omega^h \\ -.5\lambda^h_{xx}(\rho_s - 1)^2 - .5(1-\gamma)^2\Theta^h \end{bmatrix} \\
\Gamma_{m_x,\varepsilon} &= (1-\gamma)\Gamma^h_{r,\varepsilon} - \lambda^h_x M \\
\Gamma_{m_x,M\varepsilon} &= (1-\gamma)B^h_3 - .5\lambda^h_{xx} \\
\overline{m}_x &= \log(\delta^h) - .5(1-\gamma)^2\Phi^h + .5\log[\det(I - 2(1-\gamma)^2(B^i_3)^2 M\Sigma M')]
\end{aligned}
$$

Define $P^1_t \equiv \mathbf{EPV}_t(1)$, then P^1_t is characterized by the following recursion:

$$
P^1_t = E_t[\exp^{m_{x,t+1}}(1 + P^1_{t+1})]
$$

Notice that P^1_t represents the value of the tree delivering the home goods normalized by number of home goods produced. We will refer to P^1_t also as W^h_t, the wealth of the home country.

Guess that $p^1_t = \overline{p}^1 + A_1[z_{1,t} \quad z_{2,t} \quad s_t \quad s^2_t]'$ and log-linearize the right-hand side. The following holds:

$$
\begin{aligned}
A_1 &= \begin{bmatrix} 0 & 0 & \frac{\Gamma_{m_x,s(3)}}{1-\rho_s\kappa_0} & \frac{\Gamma_{m_x,s(4)}}{1-\rho_s^2\kappa_0} \end{bmatrix} \\
\kappa_0 &\equiv \frac{\overline{P}^1}{1+\overline{P}^1}
\end{aligned}
$$

At this point, κ_0 can be found solving the following non-linear equation:

$$
\kappa_0 = \frac{\exp^{\overline{m}_x + .5(\Gamma_{m_x,\varepsilon}+\kappa_0 A_{1(3)}M)\Sigma(\Gamma_{m_x,\varepsilon}+\kappa_0 A_{1(3)}M)'}}{\sqrt{\det(I - 2(\kappa_0 A_{1(4)} + \Gamma_{m_x,M\varepsilon})^2 M\Sigma M')}}
$$

Define $P_t^2 \equiv \mathbf{EPV}_t\left((1+\kappa)\frac{S_t}{1+\kappa S_t}\right)$, then P_t^2 is characterized by the following recursion:

$$
P_t^2 = E_t\left[\exp^{m_{x,t+1}}\left((1+\kappa)\frac{S_t}{1+\kappa S_t} + P_{t+1}^2\right)\right]
$$

Guess that $p_t^2 = \overline{p}^2 + A_2[z_{1,t} \quad z_{2,t} \quad s_t \quad s_t^2]'$ and log-linearize the right-hand side. The following holds:

$$
\begin{aligned}
A_2 &= \begin{bmatrix} 0 & 0 & \frac{\Gamma_{m_x,s(3)} + \frac{\kappa_1}{1+\kappa\overline{S}}\rho_s}{1-\rho_s\kappa_2} & \frac{\Gamma_{m_x,s(4)}}{1-\rho_s^2\kappa_2} \end{bmatrix} \\
\kappa_1 &\equiv \frac{(1+\kappa)\frac{\overline{S}}{1+\kappa\overline{S}}}{(1+\kappa)\frac{\overline{S}}{1+\kappa\overline{S}} + \overline{P}^2} \\
\kappa_2 &\equiv \frac{\overline{P}^2}{(1+\kappa)\frac{\overline{S}}{1+\kappa\overline{S}} + \overline{P}^2}
\end{aligned}
$$

At this point, $\overline{P}^2$ can be found solving the following non-linear equation:

$$
\begin{aligned}
\overline{P}^2 &= \left((1+\kappa)\frac{\overline{S}}{1+\kappa\overline{S}}+\overline{P}^2\right)\frac{\exp^{\overline{m}_x+.5\Gamma_P\Sigma\Gamma_P'}}{\sqrt{\det(I-2(\kappa_2A_{2(4)}+\Gamma_{m_x,M\varepsilon})^2M\Sigma M')}} \\
\Gamma_P &\equiv \Gamma_{m_x,\varepsilon}+\kappa_2A_{2(3)}M+\kappa_1M/(1+\kappa\overline{S})
\end{aligned}
$$

At the equilibrium:

$$
a_t^h = \frac{\overline{W^h}}{X}\left(e^{A_2[z_{1,t} \quad z_{2,t} \quad s_t \quad s_t^2]'}-e^{A_1[z_{1,t} \quad z_{2,t} \quad s_t \quad s_t^2]'}\right)
$$

Define:

$$
f(s_t)=e^{A_2[z_{1,t} \quad z_{2,t} \quad s_t \quad s_t^2]'}-e^{A_1[z_{1,t} \quad z_{2,t} \quad s_t \quad s_t^2]'}
$$

At the steady state $f(0)=0$. Also, notice that $f'>0$. Hence at the equilibrium:

$$
s_t = f^{-1}\underbrace{\left(\frac{A_t^h}{W_t^h}\right)}_{+}
$$