Implications of Variance Bounds on the Permanent and Transitory Components of Stochastic Discount Factors for Asset Pricing Models*

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This draft, December 2010

Abstract

We develop a lower bound on the variance of the permanent component of stochastic discount factors (SDFs), a lower (upper) bound on the size of the permanent (transitory) component of SDFs, and a lower bound on the variance of the ratio of the permanent to the transitory component of SDFs. A salient feature of our variance bounds is that they incorporate information from both average returns and the variance-covariance matrix of returns. Our treatment improves on Alvarez and Jermann (2005, Econometrica) in two other ways. First, we generalize the bounds framework to accommodate an asset space with a dimension greater than three. Second, we can accommodate a setting where some assets have a negative expected rate of return. Our variance bounds, when combined with exactly solved eigenfunction problems, can suggest improvements in asset pricing models. We apply our framework to investigate the empirical attributes of models in the long-run risk class.

KEY WORDS: Stochastic discount factors, permanent component, transitory component, variance bounds, asset pricing models, eigenfunction problem.

JEL CLASSIFICATION CODES: G11, G12, G13, C5, D24, D34.

*The authors acknowledge helpful discussions with Andrew Ang, Xiaohui Gao, Anisha Ghosh, Eric Ghysels, Steve Heston, Bryan Kelly, Pete Kyle, George Panayotov, Eric Renault, Georgios Skoulakis, Rene Stulz, Jessica Wachter, Michael Weisbach, Ingrid Werner, Wei Yang, and Lu Zhang. Seminar participants at Ohio State University provided useful suggestions. We are especially grateful for the comments and advice of a referee. We welcome comments, including references to related papers we have inadvertently overlooked. Any remaining errors are our responsibility alone.

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1. Introduction

In an important contribution, Alvarez and Jermann (2005) lay the foundations for deriving bounds when the stochastic discount factor (hereby, SDF), from an asset pricing model, can be decomposed into a permanent component and a transitory component (see also the contribution of Hansen, Heaton, and Li (2008), and Hansen and Scheinkman (2009)). The purpose of this paper is to propose a bounds methodology to assess asset pricing models, and to improve on the Alvarez and Jermann (2005) bounds.

We first ask and address two methodological questions. First, is it possible to generalize the bounds framework to accommodate an asset space with a dimension greater than three? Our rationale is that the bounds postulated in Alvarez and Jermann (2005) hinge on the return properties of the risk-free bond, the long-term discount bond, and the equity portfolio. Second, can one modify the bounds framework to a setting where some assets have a negative expected rate of return and, hence, generalize the approach?

Building on our treatment, we turn to a question of economic interest: What can be learned about asset pricing models that consistently price the risk-free bond, long-term discount bond, equity portfolios, and insurance assets (where high price is accompanied by low expected payout, yielding a negative expected rate of return), when the SDF has a permanent and a transitory component? Claims on market variance that depict insurance-like features are now actively traded, and some SDFs, such as those based on long-run risk models, incorporate a role for market variance (e.g., Bansal and Yaron (2004)). We solve the eigenfunction problem of Hansen and Scheinkman (2009) for asset pricing models in the long-run risk class, enabling us to evaluate their performance based on our bounds on the permanent and transitory components.[1]

In our setup, we develop the lower bound on the variance of the permanent component of SDFs, the lower (upper) bound on the size of the permanent (transitory) component of SDFs, and then a lower bound on the variance of the ratio of the permanent to the transitory component of SDFs. The lower bound on the variance of the permanent component of SDFs and the lower (upper) bound on the size of the permanent (transitory) component can be viewed as a generalization of the Alvarez and Jermann (2005) bounds.

Our lower bound on the variance of the ratio of the permanent to the transitory component of SDFs allows us to assess whether SDFs implied from asset pricing models are capable of describing the additional dimension of comovement between bond, equity, and other markets, and has no analog in Alvarez and Jer-

[1] Besides, our bounds framework is amenable to investigating the suitability of SDFs to address several asset pricing puzzles together, for instance, by combining elements of the value premium, the equity premium, the risk-free return, and the bond risk premium. In this regard, our work is related to, among others, Campbell, Hilscher, and Szilagyi (2008), Cochrane and Piazzesi (2005), Koijen, Lustig, and Van Nieuwerburgh (2009), Lettau and Wachter (2007), and Yang (2009, 2010).

mann (2005).[2] A salient feature of our bounds is that they incorporate information from average returns as well as the variance-covariance matrix of returns. Probing further, our analysis reveals that non-normalities in the data can bring deviations between our measures and the Alvarez and Jermann (2005) counterpart.

The posited bounds, in conjunction with an analytical solution to the eigenfunction problem, can provide a way to discern whether the time-series properties underlying an asset pricing model are consistent with observed data from financial markets. In this respect, we focus first on a long-run risk model that features a small but persistent component in consumption, as proposed in Bansal and Yaron (2004), while the second model introduces non-gaussian shocks to the consumption growth process, as proposed in Kelly (2009). Using the eigenfunction approach, we show that the transitory and permanent components of the SDF in the model of Bansal and Yaron are lognormally distributed, while the permanent component of the SDF in the model of Kelly is not lognormally distributed. Having both model types in our investigation corroborates that a broad set of distributional properties can be accommodated within our approach. The importance of studying long-run risks is also recognized by Hansen (2009).

Moving to the empirical investigation, our bounds facilitate a number of insights about the performance of long-run risk models. One implication of our findings is that the variance of the permanent component of the SDF in long-run risk models is of an order lower than the corresponding variance reflected by the returns data. Moreover, the size of the transitory component of SDFs in long-run risk models is more pronounced compared to the data counterparts, revealing a source of the limitation in characterizing the behavior of bond returns. Relevant to our findings, we develop a theoretical restriction aimed at reconciling the observation that an asset pricing model can match the equity premium but does not offer a sufficiently volatile permanent component of the SDF.

We also show that the model-based variance of the ratio of the permanent to the transitory component of the SDF is insufficiently high, conveying the need to describe more plausibly the joint movement of returns of bonds and equities. Taken all together, our bounds-based non-parametric approach highlights the dimensions of difficulty in reconciling observed asset returns under commonly adopted parameterizations of long-run risk models. Our work belongs to a list of studies that explores the implications of long-run risk models, as outlined, for example, in Bansal, Kiku, and Yaron (2009), Beeler and Campbell (2009), Ferson, Nallareddy, and Xie (2010), Hansen, Heaton, and Li (2008), and Yang (2009, 2010).

[2] The permanent (transitory) component of SDFs is useful for consistently pricing equities and insurance assets (bonds), while the ratio of the permanent to the transitory components of SDFs links the underlying markets. The importance of linking markets has been emphasized by Campbell (1986), Campbell and Ammer (1993), Fama and French (1993), Connolly, Sun, and Stivers (2005), Baele, Bekaert, and Inghelbrecht (2010), Colacito, Engle, and Ghysels (2010), and David and Veronesi (2009).

One may ask: What is the advantage of employing variance bounds to assess asset pricing models versus matching some moments of the returns data? The heart of the bounds approach is that bounds are derived under the assumption that the permanent and transitory components of SDFs correctly price a set of assets. On the other hand, the approach of matching some sample moments of the returns data may fail to internalize broader aspects of asset return dynamics. Our variance bounds could be adopted as a complementary device to examine the validity of a model, apart from matching sample moments, slope coefficients from predictive regressions, and correlations. In this sense, our bounds approach retains the flavor of the Hansen and Jagannathan (1991) bounds, and yet offers the tractability of comparing the variance of the permanent and transitory components of SDFs to those implied by the data.

The paper is organized as follows. Section 2 develops theoretical results on the bounds and investigates their relevance in the context of Alvarez and Jermann (2005) bounds. While our approach can be employed to study any SDF featuring permanent and transitory components, Section 3 derives the permanent and transitory components of the SDF for the asset pricing models of Bansal and Yaron (2004) and Kelly (2009), by solving the eigenfunction problem. Variance bounds are our yardstick for empirically evaluating aspects of long-run risk models. Conclusions are offered in Section 4.

2. Bounds on the permanent and transitory components

This section presents theoretical bounds related to the unconditional variance of the permanent and the transitory component of the SDFs, and the ratio of the permanent to the transitory components of SDFs. The lower bound on the unconditional variance of the SDFs is provided in Appendix A. We also formalize the sense in which our approach can improve on the Alvarez and Jermann (2005) bounds.

Since asset pricing models often face a hurdle of explaining asset market data based on unconditional bounds, we develop our results in terms of unconditional bounds, instead of the sharper conditional bounds. Appendix B presents the conditional variance bounds for completeness.

2.1. Consistent pricing of risk-free bond, long-term bond, equities, and insurance assets

We let $\{M_t\}$ be the process of strictly positive pricing kernels. As in Duffie (1996) and Hansen and Richards (1987), we use the absence of arbitrage opportunities to specify the current price of an asset that pays $\mathcal{H}_{t+k}$ at time $t+k$ as

$$V_t\left[\mathcal{H}_{t+k}\right] \;=\; E_t^{\mathbb{P}}\left(\frac{M_{t+k}}{M_t}\,\mathcal{H}_{t+k}\right), \tag{1}$$

where $E_t^{\mathbb{P}}(.)$ represents conditional expectation under the physical probability measure $\mathbb{P}$. The SDF is represented by $\frac{M_{t+1}}{M_t}$, and the gross asset return by $\frac{\mathcal{H}_{t+k}}{V_t[\mathcal{H}_{t+k}]}$. Under certain conditions, the physical probability measure is related to the (risk neutral) probability measure $\mathbb{Q}$ via the Radon-Nikodym representation:

$$\frac{d\mathbb{Q}}{d\mathbb{P}} \;=\; \frac{1}{V_t\left[1_{t+1}\right]}\frac{M_{t+1}}{M_t}, \quad \text{and, hence, equation (1) becomes} \quad V_t\left[\mathcal{H}_{t+1}\right] = V_t\left[1_{t+1}\right] E_t^{\mathbb{Q}}\left(\mathcal{H}_{t+1}\right), \tag{2}$$

where $E_t^{\mathbb{Q}}(.)$ is expectation under $\mathbb{Q}$.

To differentiate returns offered by different types of asset classes, we first define $R_{t+1,k}$ as the gross return from holding, from time t to $t+1$, a claim to one unit of the numeraire to be delivered at time $t+k$. Then, the return from holding a discount bond with maturity k from time t to $t+1$, and the long-term discount bond is, respectively,

$$R_{t+1,k} \;=\; \frac{V_{t+1}\left[1_{t+k}\right]}{V_t\left[1_{t+k}\right]} \qquad \text{and} \qquad R_{t+1,\infty} \;\equiv\; \lim_{k\to\infty} R_{t+1,k}. \tag{3}$$

The case of $k=1$ in equation (3) corresponds to the return of a risk-free bond, and $R_{t+1,\infty}$ corresponds to the return of a long-term discount bond.

Moving to investment assets such as equities, we define $\mathbf{R}_{t+1}^{[+]}$ as the $n_1 \times 1$ vector of gross returns. Each equity portfolio has a positive expected rate of return, net of the return of a risk-free bond.

Next consider insurance assets, where the discounted payoff under the $\mathbb{Q}$ measure dominates the $\mathbb{P}$ measure counterpart and, hence, $\frac{E_t^{\mathbb{P}}(\mathcal{H}_{t+1})}{V_t[1_{t+1}]\,E_t^{\mathbb{Q}}(\mathcal{H}_{t+1})} < 1$. Let the $n_2 \times 1$ vector of gross returns of such assets be denoted by $\mathbf{R}_{t+1}^{[-]}$.

Examples of insurance claims, considered later in our empirical exercise, include a variance swap, which has a zero cost at entry, satisfying the payoff (e.g., Carr and Wu (2009)):

$$\mathcal{VRP}_{t+1} \;\equiv\; \int_t^{t+1} \sigma_s^2\, ds \;-\; E_t^{\mathbb{Q}}\left(\int_t^{t+1} \sigma_s^2\, ds\right), \qquad \text{where} \;\; \int_t^{t+1} \sigma_s^2\, ds \equiv \mathcal{H}_{t+1}. \tag{4}$$

The variance swap is contingent on the integrated value of instantaneous market return variance σ_t^2 over a month, namely, integrated variance (e.g., Andersen, Bollerslev, Diebold, and Labys (2003)). We recognize that put options are also associated with negative expected returns in the tails. Cochrane and Saa-Requejo

(2000) derive tight bounds on asset prices that incorporate options in conjunction with other assets.

Consider the set of SDFs that consistently price the risk-free bond, the long-term discount bond, equities, and insurance assets,

$$\mathcal{S} \equiv \left\{ \frac{M_{t+1}}{M_t} : E\left(\frac{M_{t+1}}{M_t} 1_{t+1}\right) = \mu_m,\ E\left(\frac{M_{t+1}}{M_t} R_{t+1,\infty}\right) = 1,\ E\left(\frac{M_{t+1}}{M_t} \mathbf{R}_{t+1}^{[+]}\right) = \iota,\ E\left(\frac{M_{t+1}}{M_t} \mathbf{R}_{t+1}^{[-]}\right) = \iota \right\}, \tag{5}$$

where $1/\mu_m$ is the return on the risk-free bond, and ι is a vector of ones of the appropriate dimension. $E(.)$ represents the *unconditional* expectation operator with respect to the physical probability measure $\mathbb{P}$, where we have suppressed the superscript $\mathbb{P}$ for compactness (unless stated otherwise). Moreover, let $Var[.]$ and $Cov[.,.]$ be the *unconditional* variance and the covariance, respectively, under $\mathbb{P}$.

The *L*-measure-based bounds framework in Alvarez and Jermann (2005) is intended specifically for a risk-free bond, a long-term discount bond, and a risky equity portfolio, whereas equation (5) allows one to expand the asset space to a dimension beyond three, and to insurance assets.

We have two reasons to expand the set of assets to include insurance assets in our theoretical and empirical analysis, particularly, claims on market return variance. The first reason is that market variance constitutes an important traded quantity and an asset class (Andersen and Benzoni (2009), Carr and Lee (2008), and Gatheral (2006)), and extant literature has shown that the time-series assumptions on the log consumption growth process, and its conditional variance, can impose constraints on the SDF specification. For example, SDFs implied by the Bansal and Yaron (2004) model, and its extensions, suggest restrictions on the variance risk premium $\mathcal{VRP}_{t+1}$ (as posited in (4)). In such asset pricing models, the expected variance risk premium is driven by both long- and short-run risks and, hence, the permanent and transitory components of SDFs can impact the pricing of variance swaps.

The second reason is that the expected variance risk premium is negative, and we develop a bounds framework for permanent and transitory components that can be applied to assets with negative risk premiums. The viability of asset pricing models can now be judged by their ability to satisfactorily accommodate the risk premium on a spectrum of traded assets, both maximally positive and highly negative.

2.2. Bounds on the permanent component of the SDF

Alvarez and Jermann (2005, Proposition 1), Hansen, Heaton, and Li (2008), and Hansen and Scheinkman

(2009) show that any SDF can be decomposed into a permanent component and a transitory component. Inspired by their analyses, we presume that there exists a decomposition of the pricing kernel M_t into a permanent and a transitory component of the type:

$$M_t \;=\; M_t^P M_t^T, \qquad \text{with} \qquad E_t\left(M_{t+1}^P\right) \;=\; M_t^P. \tag{6}$$

The permanent component M_t^P is a martingale, while the transitory component M_t^T is a scaled long-term interest rate. In particular,

$$R_{t+1,\infty} \;=\; \frac{M_t^T}{M_{t+1}^T}. \tag{7}$$

which follows from Alvarez and Jermann (2005, Assumptions 1 and 2, and Proposition 2).

We define the $1+n_1+n_2$-dimensional vector as $\mathbf{R}_{t+1} = \left(R_{t+1,1}, \mathbf{R}_{t+1}^{[+]}, \mathbf{R}_{t+1}^{[-]}\right)$, which is the gross return vector containing the risk-free bond, the investment assets, and the insurance assets. Assume that the variance-covariance matrix of $\mathbf{R}_{t+1}$, $\mathbf{R}_{t+1}/R_{t+1,\infty}$, $\mathbf{R}_{t+1}/R_{t+1,\infty}^2$ are each nonsingular. Our proof follows.

Proposition 1 *Under the decomposition (6), the lower bound on the unconditional variance of the permanent component of SDFs $\frac{M_{t+1}}{M_t} \in \mathcal{S}$ is:*

$$Var\left[\frac{M_{t+1}^P}{M_t^P}\right] \geq \sigma_{pc}^2 \equiv \begin{pmatrix} 1 \;-\; E\left(\frac{R_{t+1,1}}{R_{t+1,\infty}}\right) \\ \iota \;-\; E\left(\frac{\mathbf{R}_{t+1}^{[+]}}{R_{t+1,\infty}}\right) \\ \iota \;-\; E\left(\frac{\mathbf{R}_{t+1}^{[-]}}{R_{t+1,\infty}}\right) \end{pmatrix}' \left(Var\left[\frac{\mathbf{R}_{t+1}}{R_{t+1,\infty}}\right]\right)^{-1} \begin{pmatrix} 1 \;-\; E\left(\frac{R_{t+1,1}}{R_{t+1,\infty}}\right) \\ \iota \;-\; E\left(\frac{\mathbf{R}_{t+1}^{[+]}}{R_{t+1,\infty}}\right) \\ \iota \;-\; E\left(\frac{\mathbf{R}_{t+1}^{[-]}}{R_{t+1,\infty}}\right) \end{pmatrix}. \tag{8}$$

Furthermore, when equation (8) holds, the ratio of the unconditional variance of the permanent component to the unconditional variance of the SDF satisfies,

$$\frac{Var\left[\frac{M_{t+1}^P}{M_t^P}\right]}{Var\left[\frac{M_{t+1}}{M_t}\right]} \;\geq\; \frac{\sigma_{pc}^2}{\sigma_{pc}^2 \;+\; 1 \;-\; \mu_m^2}, \tag{9}$$

where σ_{pc}^2 is defined in equation (8), and μ_m is the mean of the SDF.

Proof: See the Appendix A. ■

Our variance bounds are general and are not specific to the Alvarez and Jermann (2005), Hansen, Heaton, and Li (2008), or Hansen and Scheinkman (2009) decomposition. They apply to any SDF that can be decomposed into a permanent and a transitory component.

The σ^2_{pc} in inequality (8) is computable, given the return time-series of risk-free bond, long-term bond, equities, and insurance assets, for instance, variance swaps. In general, our bounds combine information from the vector of average returns and the variance-covariance matrix of returns.

Although the Hansen and Jagannathan (1991) bound was not developed to differentiate between the permanent and the transitory components of the SDF, the variance bound on the permanent component σ^2_{pc} is amenable to an interpretation, as in the Hansen and Jagannathan (1991) bound (see also Cochrane and Hansen (1992)). To appreciate this feature, we let $R^{\mathrm{vs}}_{t+1} = \frac{\int_t^{t+1} \sigma_s^2\, ds}{V_t[1_{t+1}] E_t^{\mathbb{Q}}\left(\int_t^{t+1} \sigma_s^2\, ds\right)}$ and we specialize our analysis to a three-dimensional vector of gross returns $\mathbf{R}^v_{t+1} = [R_{t+1,1}, R^{\mathrm{eq}}_{t+1}, R^{\mathrm{vs}}_{t+1}]$, consisting of a risk-free bond, the equity market, and the variance swap. Notice,

$$\frac{R^{\mathrm{eq}}_{t+1}}{R_{t+1,\infty}} \simeq 1 + \log(R^{\mathrm{eq}}_{t+1}) - \log(R_{t+1,\infty}), \qquad \text{and} \qquad \frac{R^{\mathrm{vs}}_{t+1}}{R_{t+1,\infty}} \simeq 1 + \log(R^{\mathrm{vs}}_{t+1}) - \log(R_{t+1,\infty}). \tag{10}$$

Following Hansen and Jagannathan (1991) and Cochrane (2005, Sections 5.5–5.6), the variance bound on the permanent component of SDFs can be interpreted as the maximum Sharpe ratio when the investment opportunity set is comprised of an equity market with excess return relative to the long-term bond, and the variance swap with excess return relative to the long-term bond (assuming the risk-free return is close to zero).

Inequality (9) bounds the ratio of the variance of the permanent component of the SDF to the variance of the SDF, which can be a useful object for understanding what time-series assumptions are necessary to achieve consistent risk pricing across a multitude of asset markets. In addition to using the information content of returns across different asset classes, the bounds provided in (8) and (9) are essentially model-free and can be employed to evaluate the permanent component of any SDF, regardless of its distribution.

Koijen, Lustig, and Van Nieuwerburgh (2009) highlight the economic role of $Var\left[\frac{M^P_{t+1}}{M^P_t}\right] / Var\left[\frac{M_{t+1}}{M_t}\right]$ in affine models. When both the permanent and transitory components of the SDF are lognormally distributed, they specifically show that $Var\left[\frac{M^P_{t+1}}{M^P_t}\right] / Var\left[\frac{M_{t+1}}{M_t}\right]$ implied within their model is almost perfectly correlated with the Cochrane and Piazzesi (2005) factor.

Finally, the lower bound on the variance of SDFs is derived in equation (A18) of Appendix A. There is a key difference between the lower bound on the variance of SDFs and the lower bound on the variance of the permanent component of SDFs. To be compatible with the high Sharpe ratio when the investment opportunity is comprised of risk-free bond, long-term bond, equities, and insurance assets, SDFs have to be volatile. To be compatible with the low returns of long-term bonds relative to equities, the permanent component of SDFs has to be arguably large.

2.3. *Bounds on the transitory component of the SDF*

While a central constituent of any SDF is the permanent component, a second necessary constituent is the transitory component, which equals the inverse of the return of an infinite-maturity discount bond and governs the behavior of interest rates. Absent a transitory component, the excess returns of discount bonds are zero, which contradicts empirical evidence (e.g., Fama and Bliss (1987), Campbell and Shiller (1991), and Cochrane and Piazzesi (2005)).

To gauge the ability of SDFs to explain bond market data, while consistently pricing the remaining set of assets in (5), we provide an upper bound on the variance of the transitory component of SDFs.

Proposition 2 *Under the decomposition (6), the upper bound on the ratio of the unconditional variance of the transitory component of SDF to the unconditional variance of SDF* $\frac{M_{t+1}}{M_t} \in \mathcal{S}$ *is*

$$\frac{Var\left[\frac{M_{t+1}^T}{M_t^T}\right]}{Var\left[\frac{M_{t+1}}{M_t}\right]} \leq \frac{Var\left[\frac{1}{R_{t+1,\infty}}\right]}{\begin{pmatrix} 1 - \mu_m E\left(R_{t+1,1}\right) \\ \iota - \mu_m E\left(\mathbf{R}_{t+1}^{[+]}\right) \\ \iota - \mu_m E\left(\mathbf{R}_{t+1}^{[-]}\right) \end{pmatrix}' \left(Var\left[\mathbf{R}_{t+1}\right]\right)^{-1} \begin{pmatrix} 1 - \mu_m E\left(R_{t+1,1}\right) \\ \iota - \mu_m E\left(\mathbf{R}_{t+1}^{[+]}\right) \\ \iota - \mu_m E\left(\mathbf{R}_{t+1}^{[-]}\right) \end{pmatrix}}. \tag{11}$$

Proof: See Lemma 1 in Appendix A. ■

Inequality (11) places an upper bound on the ratio of the variance of the transitory component of SDFs to the variance of SDFs. The variance of the transitory component of any SDF that consistently prices bonds should be lower than the bound in equation (11).

2.4. *Bounds on the ratio of the permanent to transitory component of the SDF*

A third necessary feature of SDFs is their ability to link the behavior of the bond market to other markets. In this regard, a construct suitable for understanding cross-market relationships is the ratio of the permanent component of the SDF to the transitory component of the SDF.

Proposition 3 *Under the decomposition (6), the lower bound on the unconditional variance of the ratio of the permanent component of the SDF to the transitory component of the SDF* $\frac{\frac{M^P_{t+1}}{M^P_t}}{\frac{M^T_{t+1}}{M^T_t}}$ *is*

$$Var\left[\frac{\frac{M^P_{t+1}}{M^P_t}}{\frac{M^T_{t+1}}{M^T_t}}\right] \geq \begin{pmatrix} 1 - \mu_{m^P/m^T} E\left(\frac{R_{t+1,1}}{R^2_{t+1,\infty}}\right) \\ \iota - \mu_{m^P/m^T} E\left(\frac{\mathbf{R}^{[+]}_{t+1}}{R^2_{t+1,\infty}}\right) \\ \iota - \mu_{m^P/m^T} E\left(\frac{\mathbf{R}^{[-]}_{t+1}}{R^2_{t+1,\infty}}\right) \end{pmatrix}' \left(Var\left[\frac{\mathbf{R}_{t+1}}{R^2_{t+1,\infty}}\right]\right)^{-1} \begin{pmatrix} 1 - \mu_{m^P/m^T} E\left(\frac{R_{t+1,1}}{R^2_{t+1,\infty}}\right) \\ \iota - \mu_{m^P/m^T} E\left(\frac{\mathbf{R}^{[+]}_{t+1}}{R^2_{t+1,\infty}}\right) \\ \iota - \mu_{m^P/m^T} E\left(\frac{\mathbf{R}^{[-]}_{t+1}}{R^2_{t+1,\infty}}\right) \end{pmatrix}, \quad (12)$$

where μ_{m^P/m^T} *is the mean of the ratio of the permanent component of the SDF to the transitory component of the SDF.*

Proof: See the Appendix A. ■

Because the permanent and the transitory components of the SDFs are not independent,

$$Cov\left[\frac{M^P_{t+1}}{M^P_t}, \left(\frac{M^T_{t+1}}{M^T_t}\right)^{-1}\right] = \mu_{m^P/m^T} - E\left(R_{t+1,\infty}\right). \quad (13)$$

Hence, in contrast to Proposition 1 (Proposition 2) that examines separately the consistent pricing of equities and insurance assets (bonds), Proposition 3 discriminates, for a given comovement value $\mu_{m^P/m^T} - E\left(R_{t+1,\infty}\right)$ implied from asset pricing models, among those SDF that are capable of describing the observed comovement of bond returns and other asset returns.

Under the assumption that μ_{m^P/m^T} is fixed, Proposition 3 provides the lower bound on the variance of the ratio of the components of the SDF, the purpose of which is to assess whether the SDFs can explain the comovement in (13). The bound in Proposition 3 is new, with no counterpart in Alvarez and Jermann (2005).

When a diagnostic test, for instance, the Hansen and Jagannathan (1991) variance bound, rejects an asset pricing model, it often fails to ascribe model failure specifically to the inadequacy of the permanent component of SDFs, the transitory component of SDFs, or to a combination of both. Thus, our framework offers the pertinent measures to investigate which dimension of the SDF can be modified, when the goal is to capture return variation in a single market or across markets. We revisit this issue when examining models in the long-run risk class by analytically solving an eigenfunction problem and then invoking our Propositions 1, 2, and 3.

Each of the bounds derived in Propositions 1, 2, and 3 are unconditional bounds. In Appendix B, we scale the returns by conditioning variables and propose bounds that incorporate conditioning information.

2.5. *Distinction from the Alvarez and Jermann (2005) bounds*

Germane to the bounds developed in Propositions 1 and 2 are two central questions: How are the bounds distinct from the corresponding bounds in Propositions 2 and 3 in Alvarez and Jermann (2005)? In what way does our treatment improve on Alvarez and Jermann (2005)?

To address these questions, we first note that Alvarez and Jermann (2005) define the L-measure of a random variable u as

$$L[u] \equiv f[E(u)] - E(f[u]), \qquad \text{with} \qquad f[u] = \log(u). \tag{14}$$

Using $L[u] = \log(E(u)) - E(\log(u))$ as a measure of volatility, i.e, $Var[u] = E\left(u^2\right) - (E(u))^2$, Alvarez and Jermann (2005) develop their bounds in terms of the L-measure. Under their characterizations, a one-to-one correspondence exists between the L-measure and the variance measure of $\log(u)$, when u is distributed lognormally, as in $L[u] = \frac{1}{2}Var[\log(u)]$.

Still, discrepancies between the two dispersion measures can get magnified under departures from lognormality, for example Kelly (2009), where neither the SDF nor the permanent component are lognormally distributed, as captured by $|L[u] - \frac{1}{2}Var[\log(u)]| > 0$. Nonetheless, even under lognormality, $L[u] \neq \frac{1}{2}Var[u]$, as can be discerned from our Example 2 (shown shortly).

The following examples further illustrate some differences in the permanent and transitory components of the SDF across models, in our treatment and in that of Alvarez and Jermann (2005).

2.5.1. *L-measure versus the variance measure in example economies*

Example 1 (Alvarez and Jermann (2005, page 1981)). Suppose the SDF is $\frac{M_{t+1}}{M_t} = \beta\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}$, and consumption growth is independently and identically distributed. In this economy, interest rates are constant, hence,

$$R_{t+1,1} \;=\; R_{t+1,\infty} \;=\; a_0 > 1, \qquad \text{which implies} \qquad \frac{M_{t+1}}{M_t} = \frac{1}{a_0}\frac{M^P_{t+1}}{M^P_t}. \tag{15}$$

The L-measure of Alvarez and Jermann implies

$$L\left[\frac{M^P_{t+1}}{M^P_t}\right] = L\left[\frac{M_{t+1}}{M_t}\right], \qquad \frac{L\left[\frac{M^P_{t+1}}{M^P_t}\right]}{L\left[\frac{M_{t+1}}{M_t}\right]} = 1, \qquad \frac{L\left[\frac{M^T_{t+1}}{M^T_t}\right]}{L\left[\frac{M_{t+1}}{M_t}\right]} = 0. \tag{16}$$

On the other hand, the variance measure implies

$$Var\left[\frac{M^P_{t+1}}{M^P_t}\right] = a_0^2\, Var\left[\frac{M_{t+1}}{M_t}\right], \qquad \frac{Var\left[\frac{M^P_{t+1}}{M^P_t}\right]}{Var\left[\frac{M_{t+1}}{M_t}\right]} = a_0^2 > 1, \qquad \frac{Var\left[\frac{M^T_{t+1}}{M^T_t}\right]}{Var\left[\frac{M_{t+1}}{M_t}\right]} = 0, \tag{17}$$

which follows from (15) and invokes $R^{-1}_{t+1,\infty} = M^T_{t+1}/M^T_t$. ♣

The gist of Example 1 is that the lower bound on the size of the permanent component in Alvarez and Jermann (2005) is $\frac{E\left(\log\left(\frac{R^{\text{eq}}_{t+1}}{R_{t+1,1}}\right)\right) - E\left(\log\left(\frac{R_{t+1,k}}{R_{t+1,1}}\right)\right)}{E\left(\log\left(\frac{R^{\text{eq}}_{t+1}}{R_{t+1,1}}\right)\right) + L\left[\frac{1}{R_{t+1,1}}\right]} = 1$, while it is $a_0^2 > 1$ based on our Proposition 1. As the SDF only has shocks to the permanent component, the upper bound on the size of the transitory component, i.e., $\frac{L\left[\frac{1}{R_{t+1,\infty}}\right]}{E\left(\log\left(\frac{R^{\text{eq}}_{t+1}}{R_{t+1,1}}\right)\right) + L\left[\frac{1}{R_{t+1,1}}\right]}$, is zero under both treatments.

Example 2 (Alvarez and Jermann (2005, page 1997)). Suppose the log of the *unnormalized* pricing kernel evolves according to an AR(1) process,

$$\log(M_{t+1}) \;=\; \log(\beta) \;+\; \varsigma \log(M_t) \;+\; \varepsilon_{t+1}, \qquad \text{where} \qquad \varepsilon_{t+1} \sim \mathcal{N}\left(0, \sigma^2_\varepsilon\right), \tag{18}$$

with $|\varsigma| < 1$. Then, the SDF is,

$$\frac{M_{t+1}}{M_t} \;=\; \exp\left(\mathcal{A}_0 \;+\; (\varsigma - 1)\sum_{j=1}^{t} \varsigma^{j-1}\varepsilon_{t-j+1} \;+\; \varepsilon_{t+1}\right), \qquad \text{where} \tag{19}$$

$$\mathcal{A}_0 \;\equiv\; \log(\beta) \;-\; \left(1 - \varsigma^t\right)\log(\beta) \;+\; (\varsigma - 1)\,\varsigma^t \log(M_0). \tag{20}$$

Equations (19)–(20) imply a risk-free return $R_{t+1,1} = \exp\left(-\mathcal{A}_0 - (\varsigma - 1)\sum_{j=1}^{t} \varsigma^{j-1}\varepsilon_{t-j+1} - \frac{1}{2}\sigma^2_\varepsilon\right)$ and log excess bond holding return $h_t[k] = \log\left(\frac{R_{t+1,k}}{R_{t+1,1}}\right) = \frac{\sigma_\varepsilon^2}{2}\left(1-\varsigma^{2(k-1)}\right)$. We can express the transitory and permanent components of the SDF for a large k as

$$\frac{M_{t+1}^T}{M_t^T} = \exp\left(-\frac{\sigma_\varepsilon^2}{2}\left(1-\varsigma^{2(k-1)}\right) + \mathcal{A}_0 + (\varsigma-1)\sum_{j=1}^{t}\varsigma^{j-1}\varepsilon_{t-j+1} + \frac{1}{2}\sigma_\varepsilon^2\right) \quad \text{and} \tag{21}$$

$$\frac{M_{t+1}^P}{M_t^P} = \exp\left(-\frac{\sigma_\varepsilon^2}{2}\varsigma^{2(k-1)} + \varepsilon_{t+1}\right). \tag{22}$$

Now,

$$L\left[\frac{M_{t+1}}{M_t}\right] = \frac{1}{2}(\varsigma-1)^2\sigma_\varepsilon^2\left(\frac{1-\varsigma^{2t}}{1-\varsigma^2}\right) + \frac{1}{2}\sigma_\varepsilon^2, \quad L\left[\frac{M_{t+1}^P}{M_t^P}\right] = \frac{1}{2}\sigma_\varepsilon^2, \quad L\left[\frac{M_{t+1}^T}{M_t^T}\right] = \frac{1}{2}(\varsigma-1)^2\sigma_\varepsilon^2\left(\frac{1-\varsigma^{2t}}{1-\varsigma^2}\right). \tag{23}$$

Further, under our treatment, the variance of the SDF is

$$Var\left[\frac{M_{t+1}}{M_t}\right] = \left(\exp\left((\varsigma-1)^2\sigma_\varepsilon^2\left(\frac{1-\varsigma^{2t}}{1-\varsigma^2}\right) + \sigma_\varepsilon^2\right) - 1\right) \times \exp\left(2\mathcal{A}_0 + (\varsigma-1)^2\sigma_\varepsilon^2\left(\frac{1-\varsigma^{2t}}{1-\varsigma^2}\right) + \sigma_\varepsilon^2\right), \tag{24}$$

the variance of the permanent component is

$$Var\left[\frac{M_{t+1}^P}{M_t^P}\right] = \left(\exp\left(\sigma_\varepsilon^2\right) - 1\right) \times \exp\left(\left(1-\varsigma^{2(k-1)}\right)\sigma_\varepsilon^2\right), \tag{25}$$

and the variance of the transitory component is

$$Var\left[\frac{M_{t+1}^T}{M_t^T}\right] = \left(\exp\left((\varsigma-1)^2\sigma_\varepsilon^2\left(\frac{1-\varsigma^{2t}}{1-\varsigma^2}\right)\right) - 1\right) \times \exp\left(\sigma_\varepsilon^2\varsigma^{2(k-1)} + 2\mathcal{A}_0 + (\varsigma-1)^2\sigma_\varepsilon^2\left(\frac{1-\varsigma^{2t}}{1-\varsigma^2}\right)\right), \tag{26}$$

and they do not coincide with the L-theoretic counterparts in (23). ♣

Figure 1 plots the L-measure and the variance measure of the permanent and the transitory component, with $\varsigma = 0.90$, $\beta = 0.998$, while setting σ_ε^2 to 0.03 or 0.40. First, a higher level of shock uncertainty σ_ε^2 translates into a higher $Var\left[\frac{M_{t+1}^P}{M_t^P}\right]$. Second, when the pricing kernel is highly persistent, a disparity between the L-measure and the variance measure can be observed under our assumed ς and across the bond maturity k. While omitted here to save on space, the basic message, namely, that there are intrinsic differences between the two dispersion measures, obtains also under conditioning information.

[Fig. 1 about here.]

2.5.2. *Under what situations the bounds based on the variance measure may be preferable?*

Before proceeding, let $\mathbf{R}^{\mathrm{e}}_{t+1} \equiv [R_{t+1,1}, R^{\mathrm{eq}}_{t+1}]$ be a two-dimensional vector of gross returns containing the risk-free bond and an equity portfolio.

We note that the bound on the size of the permanent (transitory) component in Alvarez and Jermann (2005) is derived under the assumption that $E\left(\log\left(\frac{R^{\mathrm{eq}}_{t+1}}{R_{t+1,1}}\right)\right) + L\left[\frac{1}{R_{t+1,1}}\right] > 0$, or equivalently, $E\left(\log\left(R^{\mathrm{eq}}_{t+1}\right)\right) + \log\left(E\left(\frac{1}{R_{t+1,1}}\right)\right) > 0$, such that the SDFs correctly price the risk-free bond, the long-term discount bond, and the single equity portfolio. Keeping the above in mind, the following aspects merit further discussion.

First, as can be inferred from the properties of the L-measure (see the Appendix A in Alvarez and Jermann (2005)), the bounds they derive are designed for *at most* three assets: the risk-free bond, the long-term discount bond, and the equity portfolio, with no obvious ways to generalize to the dimension of asset space beyond three.

Second, furthermore, to push an argument, if $E\left(\log\left(R^{\mathrm{eq}}_{t+1}\right)\right) < 0$, then $E\left(\log\left(R^{\mathrm{eq}}_{t+1}\right)\right) - E\left(\log\left(R_{t+1,\infty}\right)\right) < 0$. Thus, for instance, if one were to employ the risk-free bond, the long-term discount bond, and an asset with a negative expected rate of return, the lower bound on the L-measure of the permanent component of the SDF, i.e., $L\left[\frac{M^P_{t+1}}{M^P_t}\right] \geq E\left(\log\left(R^{\mathrm{eq}}_{t+1}\right)\right) - E\left(\log\left(R_{t+1,\infty}\right)\right)$, will be *negative*. This has two ramifications. One, since the L-measure of the permanent component, if it exists, of any SDF is positive, even a constant $\frac{M^P_{t+1}}{M^P_t} = 1$ would be seen as correctly pricing assets. Two, the lower (upper) bound on the size of the permanent (transitory) component of the SDF (see Propositions 2 and 3 in Alvarez and Jermann (2005)) ceases to be a well-defined object.

Even in the setting of the risk-free bond, the long-term discount bond, and the equity portfolio, some conceptual differences between the two treatments can be highlighted. For $\mathbf{R}^{\mathrm{e}}_{t+1}$, the lower bound on the variance of the permanent component of the SDF in our Proposition 1 reduces to

$$\sigma^2_{pc} = \left(\iota - E\left(\frac{\mathbf{R}^{\mathrm{e}}_{t+1}}{R_{t+1,\infty}}\right)\right)' \left(Var\left[\frac{\mathbf{R}^{\mathrm{e}}_{t+1}}{R_{t+1,\infty}}\right]\right)^{-1} \left(\iota - E\left(\frac{\mathbf{R}^{\mathrm{e}}_{t+1}}{R_{t+1,\infty}}\right)\right), \tag{27}$$

with the understanding that ι refers to a two-dimensional vector of ones. Equation (27) is the variance of

$$\frac{M^{*P}_{t+1}}{M^{*P}_t} = 1 + \left(\iota - E\left(\frac{\mathbf{R}^{\mathrm{e}}_{t+1}}{R_{t+1,\infty}}\right)\right)' \left(Var\left[\frac{\mathbf{R}^{\mathrm{e}}_{t+1}}{R_{t+1,\infty}}\right]\right)^{-1} \left(\frac{\mathbf{R}^{\mathrm{e}}_{t+1}}{R_{t+1,\infty}} - E\left(\frac{\mathbf{R}^{\mathrm{e}}_{t+1}}{R_{t+1,\infty}}\right)\right). \tag{28}$$

Our lower bound on the variance of the permanent component of the SDF is $\sigma^2_{pc} = Var\left[\frac{M^{*P}_{t+1}}{M^{*P}_t}\right]$ and, hence,

(28) can be viewed as the solution to the problem:

$$\min_{\frac{M^P_{t+1}}{M^P_t}} \quad Var\left[\frac{M^P_{t+1}}{M^P_t}\right] \quad \text{subject to} \tag{29}$$

$$E\left(\left(\frac{M^P_{t+1}}{M^P_t}\right)\left(\frac{M^T_{t+1}}{M^T_t}\right)\mathbf{R}^{\mathrm{e}}_{t+1}\right) = \iota, \qquad E\left(\frac{M^P_{t+1}}{M^P_t}\right) = 1, \qquad \frac{M^T_{t+1}}{M^T_t} = R^{-1}_{t+1,\infty}. \tag{30}$$

Regardless of the probability distribution of $\frac{M^P_{t+1}}{M^P_t}$, our results pertain to bounds on variance.

Elaborating on the above, denote $\mathbf{a}_1^{'} = (1,0)$ and $\mathbf{a}_2^{'} = (0,1)$. Under this representation, the risk-free return is $\mathbf{a}_1^{'}\mathbf{R}^{\mathrm{e}}_{t+1}$ and the equity return is $\mathbf{a}_2^{'}\mathbf{R}^{\mathrm{e}}_{t+1}$. When the set of assets includes the risk-free bond, the long-term discount bond, and the equity portfolio, the Alvarez and Jermann lower bound is the L-measure of the permanent component,

$$\min_{\frac{M^P_{t+1}}{M^P_t}} \; L\left[\frac{M^P_{t+1}}{M^P_t}\right] \;=\; E\left(\log\left(\frac{\mathbf{a}_2^{'}\mathbf{R}^{\mathrm{e}}_{t+1}}{\mathbf{a}_1^{'}\mathbf{R}^{\mathrm{e}}_{t+1}}\right)\right) \;-\; E\left(\log\left(\frac{R_{t+1,\infty}}{\mathbf{a}_1^{'}\mathbf{R}^{\mathrm{e}}_{t+1}}\right)\right) = E\left(\log\left(\frac{\mathbf{a}_2^{'}\mathbf{R}^{\mathrm{e}}_{t+1}}{R_{t+1,\infty}}\right)\right). \tag{31}$$

Yet, it is not possible to find an analytical expression for the permanent component of SDFs, namely, $\frac{\widetilde{M}^P_{t+1}}{\widetilde{M}^P_t}$ such as $L\left[\frac{\widetilde{M}^P_{t+1}}{\widetilde{M}^P_t}\right] = E\left(\log\left(\frac{\mathbf{a}_2^{'}\mathbf{R}^{\mathrm{e}}_{t+1}}{R_{t+1,\infty}}\right)\right)$. To see the reason, we use the definition of the L-measure, whereby

$$L\left[\frac{\widetilde{M}^P_{t+1}}{\widetilde{M}^P_t}\right] \;=\; \log(1) \;-\; E\left(\log\left(\frac{\widetilde{M}^P_{t+1}}{\widetilde{M}^P_t}\right)\right), \qquad \left(\text{since } E\left(\frac{\widetilde{M}^P_{t+1}}{\widetilde{M}^P_t}\right) = 1\right). \tag{32}$$

Therefore, we can at most deduce that $E\left(\log\left(\frac{\widetilde{M}^P_{t+1}}{\widetilde{M}^P_t}\right)\right) = E\left(\log\left(\frac{R_{t+1,\infty}}{\mathbf{a}_2^{'}\mathbf{R}^{\mathrm{e}}_{t+1}}\right)\right)$, and it may not be possible to recover $\frac{\widetilde{M}^P_{t+1}}{\widetilde{M}^P_t}$ analytically in terms of the vector of asset returns. Consequently, the lower bound on the L-measure of the permanent component of SDF does not satisfy the definition of the L-measure.

Our approach potentially improves on the treatment in Alvarez and Jermann (2005) bounds by deriving the unconditional lower bound on the variance of the permanent and transitory component in Propositions 1 and 2. Specifically, our treatment of bounds can accommodate an environment where some assets have a negative expected rate of return, including the out-of-the-money index put options, and it is not necessarily limited to assets with a positive expected rate of return. Additionally, our bounds incorporate the information in both the average returns and the variance-covariance matrix of asset returns, whereas the lower bound on the L-measure is operationalized through average returns.

2.6. *Lessons from a comparison with Alvarez and Jermann (2005) bounds in the data dimension*

Recapping our theoretical results so far, the first type of bounds we propose are on the variance of the permanent component of SDFs, and on the relative contribution of the variance of the permanent component to the variance of SDFs. Such bounds are beneficial for characterizing the restrictions imposed by time-series assumptions of asset pricing models. Next, we derive an upper bound on the relative contribution of the variance of the transitory component to the variance of SDFs, which is a potentially useful tool for disentangling which time-series assumptions on the consumption growth process can more aptly capture observed features of the bond market. Finally, we provide a lower bound on the variance of the relative contribution of the permanent to the transitory component of SDFs, which can establish the link between bond pricing and the pricing of other assets. Our bounds, thus, provide a set of dimensions along which one could appraise asset pricing models.

Still, some empirical questions remain with respect to observed data in the financial markets: What is gained by generalizing the L-measure–based setup in Alvarez and Jermann (2005), when applied to the risk-free bond, the long-term bond, and the equity market? In what sense do our proposed bounds quantitatively differ from Alvarez and Jermann (2005)?

To facilitate this objective, here we follow Alvarez and Jermann (2005) in the choice of three assets, the sample period of 1946:12 to 1999:12 (637 observations), as well as expressing the point estimates from the L-measure and, hence, from the variance measure in *annualized* terms. Specifically, we rely on data (http://www.econometricsociety.org/suppmat.asp?id=61&vid=73&iid=6&aid=643) on the risk-free return, a single equity return (optimal growth portfolio based on 10 CRSP Size-Decile portfolios and the equity market return), and bond yield with maturity of 25 years. The 90% confidence intervals are also reported (in square brackets), which are based on 10,000 random samples of size 637 from the data and a block bootstrap.[3]

[3]We sample the return observations in blocks of consecutive observations instead of individual observations, to account for any dependency in the returns data. The data is divided into k^* non-overlapping blocks of length l^*, where sample size $T = k^* l^*$ (e.g., Lahiri (2003)). Then, we generate $b = 10,000$ random samples of size T from the original data, where the sampling is based on 12 blocks. We use each of the generated samples to obtain the estimates of variance bounds, generically denoted by $\mathcal{V}_b$, for $b = 1, \ldots, 10,000$. Following Davison and Hinkley (1997), the 90% bootstrap confidence interval for the variance bound, $\mathcal{V}$, is obtained as $\left[2\mathcal{V} - q\left(1-\frac{\alpha}{2}\right),\ 2\mathcal{V} - q\left(\frac{\alpha}{2}\right)\right]$, where $\alpha = 0.10$, and $q\left(\frac{\alpha}{2}\right)$ is the quantile such that 500 of the bootstrap statistics $\left\{\mathcal{V}_b\right\}_{b=1,\ldots,10,000}$ are less than or equal to $q\left(\frac{\alpha}{2}\right)$.

	$L\left[\frac{M^P_{t+1}}{M^P_t}\right]$	$Var\left[\frac{M^P_{t+1}}{M^P_t}\right]$	$\frac{L\left[\frac{M^P_{t+1}}{M^P_t}\right]}{L\left[\frac{M_{t+1}}{M_t}\right]}$	$\frac{Var\left[\frac{M^P_{t+1}}{M^P_t}\right]}{Var\left[\frac{M_{t+1}}{M_t}\right]}$	$\frac{L\left[\frac{M^T_{t+1}}{M^T_t}\right]}{L\left[\frac{M_{t+1}}{M_t}\right]}$	$\frac{Var\left[\frac{M^T_{t+1}}{M^T_t}\right]}{Var\left[\frac{M_{t+1}}{M_t}\right]}$
Estimate	0.184	1.396	0.955	0.931	0.001	0.001
90% CI	[0.165, 0.205]	[1.098, 1.622]	[0.948, 0.963]	[0.919, 0.943]	[0.000, 0.001]	[0.001, 0.001]

There are two lessons that can be drawn. First, our exercise suggests that our bounds are broadly different from Alvarez and Jermann (2005). Second, instead of $L\left[\frac{M^P_{t+1}}{M^P_t}\right]$ being approximately $\frac{1}{2}Var\left[\frac{M^P_{t+1}}{M^P_t}\right]$, the discrepancy between $L\left[\frac{M^P_{t+1}}{M^P_t}\right]$ and $\frac{1}{2}Var\left[\frac{M^P_{t+1}}{M^P_t}\right]$ is large, implying that the permanent component *is far from* being distributed normally in logs. This finding essentially suggests that the skewness and kurtosis of the permanent component of SDFs may be relevant to asset pricing, features key to the generalizations of Kelly (2009) and Yang (2010).

3. Models where the SDF is a function of the market variance and contains a permanent and a transitory component

Assumptions on consumption growth, combined with Epstein and Zin (1991) recursive utility preferences, produce SDFs in the long-run risk model of Bansal and Yaron (2004) that are related to market variance.[4] A model feature relevant to our study, as we highlight, is that both the permanent and the transitory component of the SDF embed market volatility, and consistently price claims on market volatility.

Here we aim to contribute to the literature in two ways. First, we analytically solve the eigenfunction problem to determine the permanent and transitory components of the SDFs in the long-run risk class, focusing on the Bansal and Yaron (2004) model where the SDF is lognormal, and also on the Kelly (2009) model where the SDF is not lognormal. Second, we apply our variance bounds to examine how the time-series dynamics of nondurable consumption growth could be modified to improve model performance.

To be able to analytically solve the eigenfunction problems, where the SDFs are both lognormal as well as non-lognormal, can enrich the setting for researching the relevance of bounds in Propositions 1, 2, and 3,

[4]Recent studies include, among others, Bansal, Dittmar, and Kiku (2009), Bansal, Kiku, and Yaron (2009), Beeler and Campbell (2009), Constantinides and Ghosh (2008), Croce, Lettau, and Ludvigson (2008), Ferson, Nallareddy, and Xie (2010), Kaltenbrunner and Lochstoer (2010), Kelly (2009), Koijen, Lustig, Van Nieuwerburgh, and Verdelhan (2010), Lettau and Ludvigson (2004), Malloy, Moskowitz, and Vissing-Jorgensen (2009), Parker and Julliard (2005), Yang (2010), Yu (2007), and Zhou and Zhu (2009). Among the contributions on understanding the behavior of claims on market variance and market volatility, we mention, among others, Bakshi, Madan, and Panayotov (2010), Bollerslev, Sizova, and Tauchen (2009), Bollerslev, Tauchen, and Zhou (2009), Carr and Wu (2009), and Drechsler and Yaron (2009).

even though any asset pricing model featuring permanent and transitory components can be employed. Some examples include Aït-Sahalia, Parker, and Yogo (2004), Barro (2006), Bekaert and Engstrom (2010), Campbell and Cochrane (1999), Gabaix (2009), Guvenen (2009), Ju and Miao (2009), Routledge and Zin (2010), and Wachter (2009).

3.1. Eigenfunction problem in the Bansal and Yaron (2004) model, where the SDF is lognormal

To determine the permanent and transitory components of the SDF through the eigenfunction problem, we specify the evolution of (log) nondurable consumption growth g_{t+1} (Bansal and Yaron (2004)) as,

$$g_{t+1} = \mu + x_t + \sigma_t \eta_{t+1}, \qquad x_{t+1} = \rho x_t + \varphi_e \sigma_t e_{t+1}, \tag{33}$$

$$\sigma^2_{t+1} = \sigma^2 + \nu_1 \left(\sigma^2_t - \sigma^2\right) + \sigma_\omega \omega_{t+1}, \qquad \omega_{t+1},\, e_{t+1},\, \eta_{t+1} \sim i.i.d\ \mathcal{N}(0,1). \tag{34}$$

In the model, x_t is a persistently varying component of the expected consumption growth rate, and σ^2_t is the conditional variance of consumption with unconditional mean σ^2. Under the Epstein and Zin (1991) recursive utility, the time-series dynamics (33)–(34) produce a SDF that can be expressed as a function of market variance, and can be analytically separated into a permanent and a transitory component.

Proposition 4 *Solving the eigenfunction problem, the permanent and transitory components of the SDF in the long-run risk model of Bansal and Yaron (2004) are*

$$\frac{M^T_{t+1}}{M^T_t} = \delta \exp\left(-c' \left(\mathcal{X}_{t+1} - \mathcal{X}_t\right)\right) \qquad \text{and} \qquad \frac{M^P_{t+1}}{M^P_t} = \frac{M_{t+1}}{M_t} \Big/ \frac{M^T_{t+1}}{M^T_t}, \tag{35}$$

where letting $\mathcal{X}_t = \begin{pmatrix} x_t \\ \sigma^2_t - \sigma^2 \end{pmatrix}$, *and* $c' = (c_1, c_2)$ *is defined in (C32). Furthermore, the SDF* $\frac{M_{t+1}}{M_t}$ *is*

$$\frac{M_{t+1}}{M_t} = \exp\left(\zeta_t + \mathcal{D}_1 \left(g_{t+1} - E_t\left(g_{t+1}\right)\right) + \mathcal{D}_2 \left(x_{t+1} - E_t\left(x_{t+1}\right)\right) + \mathcal{D}_3 \left(\sigma^2_{M,t+1} - E_t\left(\sigma^2_{M,t+1}\right)\right)\right), \tag{36}$$

where ζ_t, $\mathcal{D}_1$, $\mathcal{D}_2$, *and* $\mathcal{D}_3$ *are defined in Appendix C. The innovation in the market variance* $\sigma^2_{M,t+1}$ *is*

$$\sigma^2_{M,t+1} - E_t\left(\sigma^2_{M,t+1}\right) = \left(1 + \kappa_1^2 A_1^2 \varphi_e^2\right) \sigma_\omega \omega_{t+1}. \tag{37}$$

Proof: See the Appendix C. ■

Equation (35) is obtained by solving the eigenfunction problem of Hansen and Scheinkman (2009):

$$E_t\left(\frac{M_{t+1}}{M_t}\frac{1}{M^e_{t+1}}\right) \;=\; \delta\frac{1}{M^e_t}, \tag{38}$$

where $M^T_t = \delta^t M^e_t$ is the transitory component of the SDF in (36), the permanent component of the SDF satisfies $M_t = M^P_t M^T_t$, and δ is the dominant eigenvalue.

In the setting of (36), $\log\left(\frac{M_{t+1}}{M_t}\right)$ is distributed normally and, hence, $\frac{M_{t+1}}{M_t}$ is distributed lognormally. The derived solution (35) is pivotal to testing long-run risk models by examining bounds on the permanent and transitory components of the SDF.

3.2. *Eigenfunction problem in the Kelly (2009) model, where the SDF is not lognormal*

Consider now Kelly (2009), who modifies the long-run risk model to include heavy-tailed shocks while maintaining the Epstein and Zin (1991) recursive utility. In this modification, non-gaussian tails in consumption growth are governed by a tail risk state variable, Λ_t.

$$g_{t+1} \;=\; \mu \,+\, x_t \,+\, \sigma_g\sigma_t z_{g,t+1} \,+\, \sqrt{\Lambda_t}\,W_{g,t+1}, \qquad x_{t+1} \;=\; \rho_x x_t \,+\, \sigma_x\sigma_t\, z_{x,t+1}, \tag{39}$$

$$\sigma^2_{t+1} \;=\; \overline{\sigma}^2(1-\rho_\sigma) \,+\, \rho_\sigma\sigma^2_t \,+\, \sigma_\sigma z_{\sigma,t+1}, \qquad \Lambda_{t+1} \;=\; \overline{\Lambda}(1-\rho_\Lambda) \,+\, \rho_\Lambda\Lambda_t \,+\, \sigma_\Lambda z_{\Lambda,t+1}, \tag{40}$$

$$z_{g,t+1},\; z_{x,t+1},\; z_{\sigma,t+1},\; z_{\Lambda,t+1} \sim i.i.d\; \mathcal{N}(0,1)\,, \qquad W_{g,t+1} \sim \text{Laplace}(0,1). \tag{41}$$

The z shocks are standard normal and independent. In addition to gaussian shocks, the consumption growth depends on non-gaussian shocks W_g, where the W_g shocks are Laplace-distributed variables with mean zero and variance 2, and independent. W_g shocks are independent of z shocks.

Proposition 5 *Solving the eigenfunction problem, the permanent and transitory components of the SDF in the model of Kelly (2009) are*

$$\frac{M^T_{t+1}}{M^T_t} \;=\; \delta\exp\left(-c^{'}\left(\mathcal{Z}_{t+1}-\mathcal{Z}_t\right)\right) \qquad \text{and} \qquad \frac{M^P_{t+1}}{M^P_t} \;=\; \frac{M_{t+1}}{M_t}\Big/\frac{M^T_{t+1}}{M^T_t}, \tag{42}$$

where letting $\mathcal{Z}_t = \begin{pmatrix} x_t \\ \sigma^2_t - \sigma^2 \\ \Lambda_t - \overline{\Lambda} \end{pmatrix}$, *and* $c^{'} = (c_1, c_2, c_3)$ *is defined in (D35)–(D37). Furthermore, the SDF*

$\frac{M_{t+1}}{M_t}$ *is*

$$\begin{aligned}\frac{M_{t+1}}{M_t} \quad = \quad & \exp\left(\xi_t + \mathcal{D}_1\left(g_{t+1}-E_t\left(g_{t+1}\right)\right) + \mathcal{D}_2\left(x_{t+1}-E_t\left(x_{t+1}\right)\right)\right. \\ & \left. + \mathcal{D}_3\left(\sigma^2_{Mt+1}-E_t\left(\sigma^2_{Mt+1}\right)\right) + \mathcal{D}_4\left(\Lambda_{t+1}-E_t\left(\Lambda_{t+1}\right)\right)\right), \end{aligned} \tag{43}$$

where ζ_t, $\mathcal{D}_1$, $\mathcal{D}_2$, $\mathcal{D}_3$, *and* $\mathcal{D}_4$ *are defined in Appendix D. The innovation in market variance* $\sigma^2_{M,t+1}$ *is*

$$\sigma^2_{M,t+1} - E_t\left(\sigma^2_{M,t+1}\right) = \lambda_\sigma\left(\sigma^2_g + \kappa_1^2 A_x^2 \sigma_x^2\right)\sigma^2_\sigma + 2\lambda_\Lambda \sigma^2_\Lambda. \tag{44}$$

Proof: See the Appendix D. ■

While the transitory component of the SDF is lognormally distributed, the permanent component of the SDF, and the SDF itself, are not lognormally distributed. Both the log of the permanent component of the SDF and the log of the SDF have non-normal distributions due to non-gaussian shocks W_g.

3.3. *The information content of variance bounds on permanent and transitory components*

Solving the eigenfunction problems, which decomposes the SDF into a permanent and a transitory component, in turn can be used to empirically evaluate models in the long-run risk class. Our thrust is to use the variance measure, and such an analysis can yield insights into how economic fundamentals are linked to SDFs, and how model performance can be enhanced by altering the properties of the permanent and transitory components of the SDFs.

Primitive parameters are chosen consistently according to the models of Bansal and Yaron (2004) and Kelly (2009), respectively, in Panels A and B of Table Appendix-I. Empirical results on the variance of the permanent and transitory components of SDFs in the long-run risk models, based on 10,000 replications, are reported in Panel A of Table 1.

Reported alongside in Panel B of Table 1 are the variance bounds obtained from two sets of data. Specifically, SET 1 contains the risk-free bond, the long-term bond, the market, and the 25 Fama-French equity portfolios sorted by size and book-to-market, while SET 2 contains the risk-free bond, the long-term bond, the market, and the 25 Fama-French equity portfolios sorted by size and momentum.

To avoid missing return observations prior to 1931:07, we employ the 1931:07 to 2009:12 sample period (942 monthly observations) to construct the bounds. The reported 90% confidence intervals (shown in square brackets) on variance bounds are obtained with a block bootstrap, when sampling is done with 15 blocks. We reiterate that tractable expressions are not available for the bounds based on the L-measure when (i) there are more than three assets, and/or (ii) at least one asset has a negative expected return.

3.3.1. *Variance of the permanent component of the SDF in long-run risk models could be enhanced*

The variance of the permanent component $Var\left[\frac{M_{t+1}^P}{M_t^P}\right]$ implied by the long-run risk model of Bansal and Yaron (2004) is 0.034, and is lower than the data-based counterparts of 0.105 (SET 1) and 0.124 (SET 2) *per month*. The impact of incorporating tails in nondurable consumption growth, as in the model of Kelly (2009), is to elevate the estimated variance of the permanent component at 0.084. Our analysis implies that the estimate of the size of the permanent component, displayed in Table 1 as $Var\left[\frac{M_{t+1}^P}{M_t^P}\right]/Var\left[\frac{M_{t+1}}{M_t}\right]$, ranges between 1.095 and 1.260.[5]

Further, even though the model of Kelly (2009) appears to describe the set of assets better, the model-based variance bound is still smaller than the lower bound on the variance of the permanent component. The p-values reported in curly brackets provide a statistical confirmation that the long-run risk models do not satisfy the lower bound on the permanent component of variance. In essence, the SDF in the long-run models could be refined to accommodate a permanent component featuring a large variance.

Pertinent to our findings, we note that each asset pricing model reasonably mimics the equity premium of about 6% and the risk-free return of about 2.5%.

In the formulation of Kelly (2009), the distributions of log SDF and the log of the permanent component of the SDF are symmetric with fat tails. From the documented results, we infer that long-run risk models could reproduce the observed variability in the permanent component of SDFs by incorporating more flexible tail properties in the SDF and in the permanent component of the SDF. We recognize nonetheless that there is insufficient evidence favoring the presence of skewness in nondurable consumption growth. Thus, an avenue to enrich long-run risk models may be to incorporate durable consumption in the dynamics of the real economy. In the spirit of our results, Yang (2010) provides evidence that durable consumption growth is left-skewed and exhibits time-varying volatility.

[5] While it may be customary to report $Var\left[\frac{M_{t+1}^P}{M_t^P}\right]/Var\left[\frac{M_{t+1}}{M_t}\right]$, it is only meaningful to adopt this measure for evaluating the contribution of the variance of the permanent component to the variance of the SDF when the permanent component of the SDF is successful, in the first stage, in explaining asset prices (as also alluded to elsewhere, for instance, Alvarez and Jermann (2005, pages 2008–2009)).

3.3.2. *The misspecified transitory component is a source of the incongruity of long-run risk models with bond market data*

Next, Table 1 reports the size of the transitory component $Var\left[\frac{M_{t+1}^T}{M_t^T}\right]/Var\left[\frac{M_{t+1}}{M_t}\right]$ implied from the long-run risk models, which is 0.071 and 0.015 in the models of Bansal and Yaron (2004) and Kelly (2009), respectively.

Both long-run risk models fail the upper bound on the transitory component, as the model-implied sizes are large compared to the data-implied entries. Thus, this part of our inquiry suggests that the adopted parameterizations may not adequately characterize the behavior of bond returns (see also Beeler and Campbell (2009)). Our approach potentially identifies a source of the misalignment of the long-run risk models with the bond market data.

3.3.3. *Models may reproduce the equity premium, but they may find it onerous to satisfy the lower bound on $Var\left[\frac{M_{t+1}^P}{M_t^P}\right]$*

Synthesizing aspects of Panels A and B of Table 1, the variance of the permanent component computed from the models is low, while the size of the transitory component is high compared to the data counterparts. Yet, the long-run risk models appear to match the equity premium and the risk-free return. This trait is also shared by the long-run risk model with inflation of Koijen, Lustig, Van Nieuwerburgh, and Verdelhan (2010), where $Var\left[\frac{M_{t+1}^P}{M_t^P}\right]$ is of the order of 0.09, similar to the model of Kelly (2009).

Why is it that an asset pricing model can come close to matching the observed equity premium, and at the same time not satisfy the lower bound on the permanent component of the SDF? Lemma 2 in Appendix E addresses this question and establishes the relevance of a bound on $Cov\left[\left(\frac{M_{t+1}^P}{M_t^P}\right)^2, \left(\frac{M_{t+1}^T}{M_t^T}\right)^2\right]$. Specifically, we derive a restriction implied by a model that explains the equity premium puzzle (i.e., the volatility of the SDF is higher than the Hansen and Jagannathan (1991) volatility bound) and produces modest variability in the permanent component of the SDF. We have verified that long-run risk models fail the restriction (E2) in Lemma 2, which potentially helps to clarify a seemingly contradictory finding.

3.3.4. *Long-run risk models cannot fully capture joint dynamics of equities and bonds*

Deserving of further comments, we now investigate the ability of long-run risk models to explain the movement of both bonds and equities. In our context, a statistic to gauge the performance of long-run risk models in capturing the comovement of returns of bonds and equities is the lower bound on $Var\left[\frac{M_{t+1}^P}{M_t^P}/\frac{M_{t+1}^T}{M_t^T}\right]$.

Table 1 shows that long-run risk models are unable to completely capture the joint movement in bond and equity markets. The variance of the ratio of the permanent component of the SDF to the transitory component of the SDF in the Bansal and Yaron (Kelly) model is estimated to be 0.045 (0.094), whereas the lower bound implied by the data is in excess of 0.138.

3.3.5. Long-run risk models do not satisfy our variance bounds under departures from standard preferences

How reasonable are the variances of the permanent and transitory components from the Bansal and Yaron (2004) model under alternative assumptions about the preference parameters, and yet calibrate to the historical levels of the equity premium and the risk-free return? How do the permanent and transitory components of the SDF change when the risk aversion and elasticity of intertemporal substitution are altered? Table 2 presents the evidence, while keeping the parameters of consumption dynamics fixed to the values in Panel A of Table Appendix-I.

We offer a few observations. First, the results show that when the risk aversion parameter γ is in the range of 10 and the elasticity of intertemporal substitution ψ ranges between 1.5 and 2.0, only then is the model able to replicate the equity premium of around 6%. When $\gamma < 7.5$, it worsens the model's ability to generate a suitable equity premium, which also coincides with low values of the variance of the permanent component of the SDF.

Further, keeping the subjective discount rate β fixed, increasing the risk aversion and the elasticity of intertemporal substitution raises the variance of the permanent component of the SDF. Our exercises also affirm that the model fails to meet the data-based bounds under reasonable combinations of (β, γ, ψ), suggesting that there may still be room to refine the dynamics of fundamentals.

3.3.6. Enforcing conformity with traded variance data can impose further hurdles on asset pricing models

Long-run risk models suggest restrictions on the variance risk premium, as reflected in the dynamics of variance swap rates, and we estimate bounds to incorporate information about the variance risk premium (as derived in Propositions 1, 2, and 3). Complementing our previous exercises, we augment SET 1 with monthly gross returns of the variance swap.[6]

Specifically, we obtain the gross return $\frac{\int_t^{t+1}\sigma_s^2\,ds}{V_t[1_{t+1}]E_t^{\mathbb{Q}}\left(\int_t^{t+1}\sigma_s^2\,ds\right)}$ by taking $\int_t^{t+1}\sigma_s^2\,ds$ to be the realized integrated variance computed from daily S&P 500 index returns, and recognizing that $E_t^{\mathbb{Q}}\left(\int_t^{t+1}\sigma_s^2\,ds\right)$ is $(\frac{\text{VIX}}{100})^2/12$ (see, e.g., Bakshi and Madan (2006), Carr and Wu (2009), and Drechsler and Yaron (2009)). The data is available over the 1990:01 to 2009:12 sample period.

		$Var\left[\frac{M_{t+1}^P}{M_t^P}\right]$	$\frac{Var\left[\frac{M_{t+1}^P}{M_t^P}\right]}{Var\left[\frac{M_{t+1}}{M_t}\right]}$	$\frac{Var\left[\frac{M_{t+1}^T}{M_t^T}\right]}{Var\left[\frac{M_{t+1}}{M_t}\right]}$	$Var\left[\frac{\frac{M_{t+1}^P}{M_t^P}}{\frac{M_{t+1}^T}{M_t^T}}\right]$
SET 1	Estimate	0.487	0.987	0.002	0.600
SET 1 augmented with variance swap	Estimate	0.543	0.989	0.001	0.662

Our results indicate that augmenting SET 1 with the variance swap magnifies the bound on the variance of the permanent component of SDFs. Indeed, such a data feature can make it difficult for asset pricing models, including the long-run risk models, to reproduce the variance of the permanent component of SDFs.

When the variance swap is a part of the set of assets, the majority of the variability in the SDFs still emanates from the permanent component. The lower bound on $Var\left[\frac{M_{t+1}^P}{M_t^P}/\frac{M_{t+1}^T}{M_t^T}\right]$ remains relatively large compared to the values under reasonable parameterizations of both the long-run risk models.

In summary, our findings highlight the difficulty of asset pricing models in explaining the permanent component of SDFs, the transitory component of SDFs, as well as the joint movement in returns across markets. A particular lesson is that asset pricing models could be enriched to better describe the dynamics of the permanent and transitory components of SDFs as reflected in bond risk premium, a positive equity risk premium, and a negative variance risk premium.

4. Conclusions

This paper proposes a framework for developing variance bounds on the permanent and transitory components of stochastic discount factors, with the aim of assessing asset pricing models. Our treatment

[6] While our focus is directed towards claims on the market variance, one could extend the analysis to include SDFs (as in equation (5)) that also consistently price out-of-the-money put options on the market index, besides primary assets. There are two salient points to appreciate in this context. First, as monthly out-of-the-money index put options often expire worthless, the index put option returns exhibit relatively high volatility compared to index returns, which can impose stringent hurdles on any asset pricing model. Second, given the third Friday of the month expiration cycle for index puts, one must resort to a data set that aligns monthly asset returns with index put option returns.

contributes by improving on Alvarez and Jermann (2005), whereby we first expand the analysis by characterizing stochastic discount factors, with permanent and transitory components, that are capable of correctly pricing the risk-free bond, long-term discount bond, equities, and insurance assets. Next, we show that our variance bounds apply broadly to a setting where some assets have a negative expected rate of return. Furthermore, our variance bounds exhibit the attribute that they combine the information from both average returns and the variance-covariance matrix of returns. Lastly, we solve the eigenfunction problem of Hansen and Scheinkman (2009) for a class of long-run risk models to evaluate their performance based on our variance bounds.

We develop theoretical results related to the lower bound on the variance of the permanent component of stochastic discount factors, the lower (upper) bound on the size of the permanent (transitory) component of stochastic discount factors, and then a lower bound on the variance of the ratio of the permanent to the transitory component. The lower bound on the variance of the permanent component of stochastic discount factors and the lower (upper) bound on the size of the permanent (transitory) component of stochastic discount factors are shown to generalize the Alvarez and Jermann (2005) bounds. Our lower bound on the variance of the ratio of the permanent to the transitory component of stochastic discount factors can be used to assess whether a candidate stochastic discount factor is capable of describing the additional dimension of comovement between bonds, equities, and other assets and has no analog in Alvarez and Jermann (2005).

The information content of our variance bounds is assessed by applying them to examine empirically the long-run risk models of Bansal and Yaron (2004) and Kelly (2009), where the stochastic discount factor and the permanent component of the stochastic discount factor is lognormal and outside of the lognormal class, respectively. The variance bounds framework developed in our paper can suggest directions in which long-run risk models could be modified to deliver further empirical consistency. In addition, the variance bounds can be employed to examine other stochastic discount factors featuring permanent and transitory components.

Our work could be extended. In particular, requiring an asset pricing model to comply with the bound restrictions imposed by the returns data could offer an alternative way to recover parameters of preferences together with those governing fundamentals. Achieving identification through the information contained in the low frequency ingredients of stochastic discount factors could lead to a better understanding of asset returns.

References

Aït-Sahalia, Y., Parker, J., Yogo, M., 2004. Luxury goods and the equity premium. Journal of Finance 59, 2959–3004.

Alvarez, F., Jermann, U., 2005. Using asset prices to measure the persistence of the marginal utility of wealth. Econometrica 73, 1977–2016.

Andersen, T., Benzoni, L., 2009. Realized volatility. Unpublished working paper. Northwestern University and Federal Reserve Bank of Chicago.

Andersen, T., Bollerslev, T., Diebold, F., Labys, P., 2003. Modeling and forecasting realized volatility. Econometrica 71, 529–626.

Baele, L., Bekaert, G., Inghelbrecht, K., 2010. The determinants of stock and bond return comovements. Review of Financial Studies 23, 2374–2428.

Bakshi, G., Madan, D., 2006. A theory of volatility spreads. Management Science 52, 1945–1956.

Bakshi, G., Madan, D., Panayotov, G., 2010. An asset pricing theory of volatility tail behavior. Unpublished working paper. University of Maryland and Georgetown University.

Bansal, R., Dittmar, R., Kiku, D., 2009. Cointegration and consumption risks in asset returns. Review of Financial Studies 22, 1343–1375.

Bansal, R., Kiku, D., Yaron, A., 2009. A note on the economics and statistics of predictability: A long run risks perspective. Unpublished working paper. Duke University and University of Pennslvania.

Bansal, R., Yaron, A., 2004. Risks for the long run: A potential resolution of asset pricing puzzles. Journal of Finance 59, 1481–1509.

Barro, R., 2006. Rare disasters and asset markets in the twentieth century. Quarterly Journal of Economics 121, 823–866.

Beeler, J., Campbell, J., 2009. The long-run risks model and aggregate asset prices: An empirical assessment. Unpublished working paper. Harvard University.

Bekaert, G., Engstrom, E., 2010. Asset return dynamics under bad environment-good environment fundamentals. Unpublished working paper. Columbia University and Federal Reserve Board.

Bollerslev, T., Sizova, N., Tauchen, G., 2009. Volatility in equilibrium: Asymmetries and dynamic dependencies. Unpublished working paper. Duke University.

Bollerslev, T., Tauchen, G., Zhou, H., 2009. Expected stock returns and variance risk premia. Review of Financial Studies 22, 4463–4493.

Campbell, J., 1986. Bond and stock returns in a simple exchange model. Quarterly Journal of Economics 101, 785–804.

Campbell, J., Ammer, J., 1993. What moves the stock and bond markets? A variance decomposition for long-term asset returns. Journal of Finance 48, 3–37.

Campbell, J., Cochrane, J., 1999. Force of habit: A consumption-based explanation of aggregate stock market behavior. Journal of Political Economy 107, 205–251.

Campbell, J., Hilscher, J., Szilagyi, J., 2008. In search of distress risk. Journal of Finance 63, 2899–2939.

Campbell, J., Shiller, R., 1991. Yield spreads and interest rate movements: A bird's eye view. Review of Economic Studies 58, 495–514.

Carr, P., Lee, R., 2008. Robust replication of volatility derivatives. Unpublished working paper. New York University and University of Chicago.

Carr, P., Wu, L., 2009. Variance risk premiums. Review of Financial Studies 22, 1311–1341.

Cochrane, J., 2005. Asset Pricing. Princeton University Press, Princeton, NJ.

Cochrane, J., Hansen, L., 1992. Asset pricing exploration for macroeconomics. In: Blanchard, O., Fischer, S. (Eds.), NBER Macroeconomics Annual. MIT Press, Cambridge, MA.

Cochrane, J., Piazzesi, M., 2005. Bond risk premia. American Economic Review 95, 138–160.

Cochrane, J., Saa-Requejo, J., 2000. Beyond arbitrage: Good-deal asset price bounds in incomplete markets. Journal of Political Economy 108, 79–119.

Colacito, R., Engle, R., Ghysels, E., 2010. A component model for dynamic correlations. Journal of Econometrics (forthcoming).

Connolly, R., Sun, L., Stivers, C., 2005. Stock market uncertainty and the stock-bond return relation. Journal of Financial and Quantitative Analysis 40, 161–194.

Constantinides, G., Ghosh, A., 2008. Asset pricing tests with long run risks in consumption growth. Unpublished working paper. NBER # 14543.

Croce, M., Lettau, M., Ludvigson, S., 2008. Investor information, long-run risks, and the duration of risky cash-flows. Unpublished working paper. NBER # 12191.

David, A., Veronesi, P., 2009. What ties return volatilities to price valuations and fundamentals?. Unpublished working paper. University of Calgary and University of Chicago.

Davison, A., Hinkley, D., 1997. Bootstrap Methods and Their Application. Cambridge University Press, Cambridge.

Drechsler, I., Yaron, A., 2009. What's vol got to do with it. Unpublished working paper. New York University and University of Pennsylvania.

Duffie, D., 1996. Dynamic Asset Pricing Theory. Princeton University Press, Princeton, NJ.

Epstein, L., Zin, S., 1991. Substitution, risk aversion, and the temporal behavior of consumption and asset returns: An empirical analysis. Journal of Political Economy 99, 263–286.

Fama, E., Bliss, R., 1987. The information in long-maturity forward rates. American Economic Review 77, 680–692.

Fama, E., French, K., 1993. Common risk factors in the returns on stocks and bonds. Journal of Financial Economics 33, 3–56.

Ferson, W., Nallareddy, S., Xie, B., 2010. The "out-of-sample" performance of long-run risk models. Unpublished working paper. University of Southern California.

Gabaix, X., 2009. Variable rare disasters: An exactly solved framework for ten puzzles in macrofinance. Unpublished working paper. New York University.

Gatheral, J., 2006. The Volatility Surface: A Practitioners Guide. John Wiley and Sons, New York, NY.

Guvenen, F., 2009. A parsimonious macroeconomic model for asset pricing. Econometrica 77, 1711–1750.

Hansen, L., 2009. Modeling the long run: Valuation in dynamic stochastic economies. Unpublished working paper. University of Chicago.

Hansen, L., Heaton, J., Li, N., 2008. Consumption strikes back? Measuring long-run risk. Journal of Political Economy 116(2), 260–302.

Hansen, L., Jagannathan, R., 1991. Implications of security market data for dynamic economies. Journal of Political Economy 99, 225–261.

Hansen, L., Richards, S., 1987. The role of conditioning information in deducing testable restrictions implied by dynamic asset pricing models. Econometrica 55, 587–613.

Hansen, L., Scheinkman, J., 2009. Long-term risk: An operator approach. Econometrica 77(2), 177–234.

Ju, N., Miao, J., 2009. Ambiguity, learning, and asset returns. Unpublished working paper. Hong Kong University of Science and Technology.

Kaltenbrunner, G., Lochstoer, L., 2010. Long-run risk through consumption smoothing. Review of Financial Studies 23, 3190–3224.

Kelly, B., 2009. Risk premia and the conditional tails of stock returns. Unpublished working paper. Booth School of Business, University of Chicago.

Koijen, R., Lustig, H., Van Nieuwerburgh, S., 2009. The cross-section and time-series of stock and bond returns. Unpublished working paper. University of Chicago, University of California-Los Angeles, and New York University.

Koijen, R., Lustig, H., Van Nieuwerburgh, S., Verdelhan, A., 2010. Long-run risk, the wealth-consumption ratio, and the temporal pricing of risk. American Economic Review (Papers and Proceedings) 100 (2), 552–556.

Lahiri, S., 2003. Resampling Methods for Dependent Data. Springer-Verlag, New York, NY.

Lettau, M., Ludvigson, S., 2004. Understanding trend and cycle in asset values: Reevaluating the wealth effect on consumption. American Economic Review 94, 276–299.

Lettau, M., Wachter, J., 2007. Why is long-horizon equity less risky? A duration-based explanation of the value premium. Journal of Finance 62, 55–92.

Malloy, C., Moskowitz, T., Vissing-Jorgensen, A., 2009. Long-run stockholder consumption risk and asset returns. Journal of Finance 64, 2427–2479.

Parker, J., Julliard, C., 2005. Consumption risk and the cross section of expected returns. Journal of Political Economy 115, 185–222.

Roman, S., 2007. Advanced Linear Algebra. Springer Verlag, London.

Routledge, B., Zin, S., 2010. Generalized disappointment aversion and asset prices. Journal of Finance 65, 1303–1331.

Wachter, J., 2009. Can time-varying risk of rare disasters explain aggregate stock market volatility?. Unpublished working paper. University of Pennsylvania.

Yang, W., 2009. Long-run risk in leisure and consumption. Unpublished working paper. University of Rochester.

Yang, W., 2010. Asset pricing with left-skewed long-run risk in durable consumption. Unpublished working paper. University of Rochester.

Yu, J., 2007. The long and the short of asset prices: Using long run consumption-return correlations to test asset pricing models. Unpublished working paper. University of Minnesota.

Zhou, G., Zhu, Y., 2009. A long-run risks model with long and short-run volatilities: Explaining predictability and volatility risk premium. Unpublished working paper. Washington University.

Appendix A: Proofs of unconditional bounds

In the results that follow, we provide a proof of Propositions 1, 2, and 3.

Proof of Proposition 1. Let $\mu_m \equiv E\left(\frac{M_{t+1}}{M_t} 1_{t+1}\right)$ and recall that $\mathbf{R}_{t+1} = \left(R_{t+1,1}, \mathbf{R}_{t+1}^{[+]}, \mathbf{R}_{t+1}^{[-]}\right)$ is the vector of gross returns with dimension $(1+n_1+n_2)$. Define the $(1+n_1+n_2)$-dimensional vector $\mathbf{A} \equiv [1, \iota, \iota]$, where ι is a vector of ones.

The proof is by construction. Start with the expression $E\left(\left(\frac{M_{t+1}^P}{M_t^P} - E\left(\frac{M_{t+1}^P}{M_t^P}\right)\right)\left(\frac{\mathbf{R}_{t+1}}{R_{t+1,\infty}} - E\left(\frac{\mathbf{R}_{t+1}}{R_{t+1,\infty}}\right)\right)\right)$. Then, noting that $E\left(\frac{M_{t+1}^P}{M_t^P}\right) = 1$ and $R_{t+1,\infty} = M_t^T / M_{t+1}^T$, we recognize

$$E\left(\left(\frac{M_{t+1}^P}{M_t^P} - E\left(\frac{M_{t+1}^P}{M_t^P}\right)\right)\left(\frac{\mathbf{R}_{t+1}}{R_{t+1,\infty}} - E\left(\frac{\mathbf{R}_{t+1}}{R_{t+1,\infty}}\right)\right)\right) \;=\; \mathbf{A} \;-\; E\left(\frac{\mathbf{R}_{t+1}}{R_{t+1,\infty}}\right). \tag{A1}$$

Denote

$$\Omega \equiv E\left(\left(\frac{\mathbf{R}_{t+1}}{R_{t+1,\infty}} - E\left(\frac{\mathbf{R}_{t+1}}{R_{t+1,\infty}}\right)\right)'\left(\frac{\mathbf{R}_{t+1}}{R_{t+1,\infty}} - E\left(\frac{\mathbf{R}_{t+1}}{R_{t+1,\infty}}\right)\right)\right) \quad \text{and} \quad \mathbf{B} \equiv \mathbf{A} - E\left(\frac{\mathbf{R}_{t+1}}{R_{t+1,\infty}}\right). \tag{A2}$$

Multiply the right hand side of (A1) by $\Omega^{-1}\mathbf{B}'$ to obtain:

$$\begin{aligned}
\mathbf{B}\Omega^{-1}\mathbf{B}' \;&=\; E\left(\left(\frac{M_{t+1}^P}{M_t^P} - E\left(\frac{M_{t+1}^P}{M_t^P}\right)\right)\left(\frac{\mathbf{R}_{t+1}}{R_{t+1,\infty}}\Omega^{-1}\mathbf{B}' - E\left(\frac{\mathbf{R}_{t+1}}{R_{t+1,\infty}}\Omega^{-1}\mathbf{B}'\right)\right)\right), &\text{(A3)}\\
&=\; Cov\left[\frac{M_{t+1}^P}{M_t^P} - E\left(\frac{M_{t+1}^P}{M_t^P}\right), \frac{\mathbf{R}_{t+1}}{R_{t+1,\infty}}\Omega^{-1}\mathbf{B}' - E\left(\frac{\mathbf{R}_{t+1}}{R_{t+1,\infty}}\Omega^{-1}\mathbf{B}'\right)\right], &\text{(A4)}\\
&\leq\; \left(Var\left[\frac{M_{t+1}^P}{M_t^P}\right]\right)^{1/2} \times \left(Var\left[\frac{\mathbf{R}_{t+1}}{R_{t+1,\infty}}\Omega^{-1}\mathbf{B}'\right]\right)^{1/2}. &\text{(A5)}
\end{aligned}$$

Given that $Var\left[\frac{\mathbf{R}_{t+1}}{R_{t+1,\infty}}\Omega^{-1}\mathbf{B}'\right]$ equals $\mathbf{B}\Omega^{-1}\mathbf{B}'$, our application of the Cauchy-Schwartz inequality implies that the lower bound on the permanent component of SDFs is

$$\sigma_{pc}^2 \;\equiv\; \mathbf{B}\Omega^{-1}\mathbf{B}' \;\leq\; Var\left[\frac{M_{t+1}^P}{M_t^P}\right]. \tag{A6}$$

Thus, we have proved the bound in equation (8) of Proposition 1. Furthermore,

$$Var\left[\frac{M_{t+1}}{M_t}\right] = Var\left[\frac{M_{t+1}^P}{M_t^P}\frac{M_{t+1}^T}{M_t^T}\right] = E\left(\left(\frac{M_{t+1}^P}{M_t^P}\right)^2\left(\frac{1}{R_{t+1,\infty}}\right)^2\right) \;-\; \mu_m^2. \tag{A7}$$

Observe that $Var\left[\frac{M_{t+1}}{M_t}\right]+\mu_m^2 = E\left(\left(\frac{M^P_{t+1}}{M^P_t}\right)^2\left(\frac{1}{R_{t+1,\infty}}\right)^2\right) \leq E\left(\left(\frac{M^P_{t+1}}{M^P_t}\right)^2\right)$. Proceeding, $Var\left[\frac{M_{t+1}}{M_t}\right]+\mu_m^2 \leq E\left(\left(\frac{M^P_{t+1}}{M^P_t}\right)^2\right)-1+1$. Therefore,

$$Var\left[\frac{M_{t+1}}{M_t}\right] \;\; \leq \;\; Var\left[\frac{M^P_{t+1}}{M^P_t}\right] \; + \; 1 \; - \; \mu_m^2. \tag{A8}$$

Expression (A8) obtains because $E\left(M^P_{t+1}/M^P_t\right)=1$. Inequality (A8) can be expressed as

$$\frac{1}{Var\left[\frac{M_{t+1}}{M_t}\right]} \geq \frac{1}{Var\left[\frac{M^P_{t+1}}{M^P_t}\right] \; + \; 1 \; - \; \mu_m^2}, \tag{A9}$$

and

$$\frac{Var\left[\frac{M^P_{t+1}}{M^P_t}\right]}{Var\left[\frac{M_{t+1}}{M_t}\right]} \geq \frac{Var\left[\frac{M^P_{t+1}}{M^P_t}\right]}{Var\left[\frac{M^P_{t+1}}{M^P_t}\right] \; + \; 1 \; - \; \mu_m^2}. \tag{A10}$$

Now the first derivative of the function $\mathcal{L}\left[u\right]=\frac{u}{u+1-\mu_m^2}$ is $\mathcal{L}'\left[u\right]=\frac{1-\mu_m^2}{(u+1-\mu_m^2)^2}>0$ because $1>\mu_m^2$. Since $Var\left[\frac{M^P_{t+1}}{M^P_t}\right] \geq \sigma^2_{pc}$, it follows that $\mathcal{L}\left[Var\left[\frac{M^P_{t+1}}{M^P_t}\right]\right] \geq \mathcal{L}\left[\sigma^2_{pc}\right]$. Hence,

$$\frac{Var\left[\frac{M^P_{t+1}}{M^P_t}\right]}{Var\left[\frac{M_{t+1}}{M_t}\right]} \;\geq\; \frac{Var\left[\frac{M^P_{t+1}}{M^P_t}\right]}{Var\left[\frac{M^P_{t+1}}{M^P_t}\right] \; + \; 1 \; - \; \mu_m^2} \;\geq\; \frac{\sigma^2_{pc}}{\sigma^2_{pc} \; + \; 1 \; - \; \mu_m^2}, \tag{A11}$$

which completes the proof. ■

Proof of Proposition 2. For the proof, we first build Lemma 1.

Lemma 1 *The lower bound on the variance of SDFs that correctly price the assets according to (5) is*

$$\sigma^2_L \;\equiv\; \begin{pmatrix} 1 \; - \; \mu_m E\left(R_{t+1,1}\right) \\ \iota \; - \; \mu_m E\left(\mathbf{R}^{[+]}_{t+1}\right) \\ \iota \; - \; \mu_m E\left(\mathbf{R}^{[-]}_{t+1}\right) \end{pmatrix}' \left(Var\left[\mathbf{R}_{t+1}\right]\right)^{-1} \begin{pmatrix} 1 \; - \; \mu_m E\left(R_{t+1,1}\right) \\ \iota \; - \; \mu_m E\left(\mathbf{R}^{[+]}_{t+1}\right) \\ \iota \; - \; \mu_m E\left(\mathbf{R}^{[-]}_{t+1}\right) \end{pmatrix}. \tag{A12}$$

Proof: Keeping the same notation as before, note that

$$E\left(\frac{M_{t+1}}{M_t} \; - \; \mu_m\right)\left(\mathbf{R}_{t+1} \; - \; E\left(\mathbf{R}_{t+1}\right)\right) = \mathbf{A} \; - \; \mu_m E\left(\mathbf{R}_{t+1}\right). \tag{A13}$$

Equation (A13) reduces to

$$E\left(\frac{M_{t+1}}{M_t} - \mu_m\right)(\mathbf{R}_{t+1} - E(\mathbf{R}_{t+1})) = \mathbf{V}, \quad \text{where defining} \quad \mathbf{V} \equiv \mathbf{A} - \mu_m E(\mathbf{R}_{t+1}). \tag{A14}$$

Multiply (A14) by $\Sigma^{-1}\mathbf{V}'$, where Σ represents the variance-covariance matrix of $\mathbf{R}_{t+1}$, that is,

$$\Sigma \equiv E(\mathbf{R}_{t+1} - E(\mathbf{R}_{t+1}))'(\mathbf{R}_{t+1} - E(\mathbf{R}_{t+1})). \tag{A15}$$

Then,

$$E\left(\frac{M_{t+1}}{M_t} - \mu_m\right)(\mathbf{R}_{t+1} - E(\mathbf{R}_{t+1}))\Sigma^{-1}\mathbf{V}' = \mathbf{V}\Sigma^{-1}\mathbf{V}'. \tag{A16}$$

Applying the Cauchy Schwartz inequality to the left hand side of (A16),

$$\mathbf{V}\Sigma^{-1}\mathbf{V}' = E\left(\frac{M_{t+1}}{M_t} - \mu_m\right)(\mathbf{R}_{t+1} - E(\mathbf{R}_{t+1}))\Sigma^{-1}\mathbf{V}' \leq \left(E\left(\frac{M_{t+1}}{M_t} - \mu_m\right)^2\right)^{\frac{1}{2}}\left(\mathbf{V}\Sigma^{-1}\mathbf{V}'\right)^{\frac{1}{2}}, \tag{A17}$$

which reduces to

$$\mathbf{V}\Sigma^{-1}\mathbf{V}' \leq E\left(\frac{M_{t+1}}{M_t} - \mu_m\right)^2 = Var\left[\frac{M_{t+1}}{M_t}\right]. \tag{A18}$$

This ends the proof of (A12).

For the σ_L^2 defined in (A12), it follows that,

$$\frac{1}{\sigma_L^2} \geq \frac{1}{Var\left[\frac{M_{t+1}}{M_t}\right]}, \qquad \text{and therefore,} \qquad \frac{Var\left[\frac{M_{t+1}^T}{M_t^T}\right]}{Var\left[\frac{M_{t+1}}{M_t}\right]} \leq \frac{Var\left[\frac{1}{R_{t+1,\infty}}\right]}{\sigma_L^2}. \tag{A19}$$

Hence, we have established the result. ∎

Proof of Proposition 3: Observe that

$$E\left(\left(\frac{\frac{M_{t+1}^P}{M_t^P}}{\frac{M_{t+1}^T}{M_t^T}} - E\left(\frac{\frac{M_{t+1}^P}{M_t^P}}{\frac{M_{t+1}^T}{M_t^T}}\right)\right)\left(\frac{\mathbf{R}_{t+1}}{R_{t+1,\infty}^2} - E\left(\frac{\mathbf{R}_{t+1}}{R_{t+1,\infty}^2}\right)\right)\right) = \mathbf{A} - \mu_{m^P/m^T} E\left(\frac{\mathbf{R}_{t+1}}{R_{t+1,\infty}^2}\right), \tag{A20}$$

where $\mu_{m^P/m^T} \equiv E\left(\frac{\frac{M^P_{t+1}}{M^P_t}}{\frac{M^T_{t+1}}{M^T_t}}\right)$. We denote

$$\Theta \equiv E\left(\left(\frac{\mathbf{R}_{t+1}}{R^2_{t+1,\infty}} - E\left(\frac{\mathbf{R}_{t+1}}{R^2_{t+1,\infty}}\right)\right)^{'}\left(\frac{\mathbf{R}_{t+1}}{R^2_{t+1,\infty}} - E\left(\frac{\mathbf{R}_{t+1}}{R^2_{t+1,\infty}}\right)\right)\right) \quad \text{and} \quad \mathbf{D} \equiv \mathbf{A} - \mu_{m^P/m^T} E\left(\frac{\mathbf{R}_{t+1}}{R^2_{t+1,\infty}}\right). \tag{A21}$$

Multiplying (A20) by $\Theta^{-1}\mathbf{D}^{'}$ and then applying the Cauchy-Schwartz inequality to the left hand side of (A20), the lower bound on the variance of the relative contribution of the permanent component of SDFs to the transitory component of SDFs can then be derived as: $\mathbf{D}\,\Theta^{-1}\,\mathbf{D}^{'} \leq Var\left[\frac{\frac{M^P_{t+1}}{M^P_t}}{\frac{M^T_{t+1}}{M^T_t}}\right]$, as asserted. ■

Appendix B: Proofs of unconditional bounds that incorporate conditioning information

Proof of Propositions 1 and 2 with conditioning variables: For tractability of exposition, we denote

$$z_t = \begin{pmatrix} z_{1t} \\ z_{2t} \\ z_{3t} \end{pmatrix} \tag{B1}$$

the set of conditioning variables. The variable z_t predicts the return's vector $\mathbf{R}_{t+1}$. We note that

$$\begin{aligned} E\left(\left(\frac{M^P_{t+1}}{M^P_t} - E\left(\frac{M^P_{t+1}}{M^P_t}\right)\right)\left(\frac{z_t^{'}\mathbf{R}_{t+1}}{R_{t+1,\infty}} - E\left(\frac{z_t^{'}\mathbf{R}_{t+1}}{R_{t+1,\infty}}\right)\right)\right) &= E\left(E_t\left(\frac{M_{t+1}}{M_t} z_t^{'}\mathbf{R}_{t+1}\right)\right) - E\left(\frac{z_t^{'}\mathbf{R}_{t+1}}{R_{t+1,\infty}}\right), \\ &= E\left(z_t^{'}\iota\right) - E\left(\frac{z_t^{'}\mathbf{R}_{t+1}}{R_{t+1,\infty}}\right). \end{aligned} \tag{B2}$$

Equation (B2) follows, since the permanent component of the pricing kernel is a martingale. Now, denote

$$\Omega \equiv E\left(\frac{z_t^{'}\mathbf{R}_{t+1}}{R_{t+1,\infty}} - E\left(\frac{z_t^{'}\mathbf{R}_{t+1}}{R_{t+1,\infty}}\right)\right)^2 \quad \text{and} \quad \mathbf{H} \equiv E\left(z_t^{'}\iota\right) - E\left(\frac{z_t^{'}\mathbf{R}_{t+1}}{R_{t+1,\infty}}\right). \tag{B3}$$

Multiplying (B2) by $\Omega^{-1}\mathbf{H}^{'}$ and applying the Cauchy-Schwartz inequality to (B2), we obtain the lower bound on the permanent component as

$$\mathbf{H}\Omega^{-1}\mathbf{H}^{'} \leq Var\left[\frac{M_{t+1}^{P}}{M_{t}^{P}}\right]. \tag{B4}$$

We then follow the same approach as in the proofs of Propositions 1 and 2 and derive the bound on the ratio of the variance of the permanent component to the variance of the SDF. ■

Proof of Proposition 3 with conditioning variables: Using the conditioning variable z_t, we observe that

$$E\left(\left(\frac{\frac{M_{t+1}^{P}}{M_{t}^{P}}}{\frac{M_{t+1}^{T}}{M_{t}^{T}}} - E\left(\frac{\frac{M_{t+1}^{P}}{M_{t}^{P}}}{\frac{M_{t+1}^{T}}{M_{t}^{T}}}\right)\right)\left(\frac{z_{t}^{'}\mathbf{R}_{t+1}}{R_{t+1,\infty}^{2}} - E\left(\frac{z_{t}^{'}\mathbf{R}_{t+1}}{R_{t+1,\infty}^{2}}\right)\right)\right) \quad = \quad E\left(z_{t}^{'}\iota\right) - \mu_{m^{P}/m^{T}}E\left(\frac{z_{t}^{'}\mathbf{R}_{t+1}}{R_{t+1,\infty}^{2}}\right). \tag{B5}$$

We denote

$$\Theta \equiv E\left(\frac{z_{t}^{'}\mathbf{R}_{t+1}}{R_{t+1,\infty}^{2}} - E\left(\frac{z_{t}^{'}\mathbf{R}_{t+1}}{R_{t+1,\infty}^{2}}\right)\right)^{2} \qquad \text{and} \qquad \mathbf{C} \equiv E\left(z_{t}^{'}\iota\right) - \mu_{m^{P}/m^{T}}E\left(\frac{z_{t}^{'}\mathbf{R}_{t+1}}{R_{t+1,\infty}^{2}}\right). \tag{B6}$$

We multiply (B5) by $\Theta^{-1}\mathbf{C}^{'}$ and apply the Cauchy-Schwartz inequality to (B5), thereby deriving the lower bound on the permanent component:

$$\mathbf{C}\Theta^{-1}\mathbf{C}^{'} \leq Var\left[\frac{\frac{M_{t+1}^{P}}{M_{t}^{P}}}{\frac{M_{t+1}^{T}}{M_{t}^{T}}}\right]. \tag{B7}$$

Thus, the result is proved. ■

Appendix C: Solution to the eigenfunction problem in Bansal and Yaron (2004)

The stochastic discount factor. Since the expressions are derived using standard techniques, we intend to be brief. The return on a consumption claim can be approximated as

$$\log\left(R_{M,t+1}\right) \;=\; \kappa_0 \;+\; \kappa_1 z_{t+1} \;+\; g_{t+1} \;-\; z_t, \tag{C1}$$

with:

$$\kappa_0 = \log\left(1+e^{\bar{z}}\right) - \frac{e^{\bar{z}}}{(1+e^{\bar{z}})}\bar{z} \qquad \text{and} \qquad \kappa_1 = \frac{e^{\bar{z}}}{(1+e^{\bar{z}})}, \tag{C2}$$

where $z_t \equiv \log\left(V_t^c/C_t\right)$ is the log price-consumption ratio. The approximate solution for the log price-consumption ratio is

$$z_t \;=\; A_0 \;+\; A_1 x_t \;+\; A_2 \sigma_t^2, \tag{C3}$$

with:

$$A_1 \;=\; \frac{\frac{1}{\psi}-1}{\kappa_1\rho-1}, \qquad A_2 \;=\; \frac{\frac{1}{2}\left(\kappa_1^2 A_1^2 \varphi_e^2 + \left(1-\frac{1}{\psi}\right)^2\right)\theta}{\left(1-\kappa_1\nu_1\right)}, \quad \text{and,} \tag{C4}$$

$$A_0 \;=\; \frac{\theta\log(\beta)+\theta\kappa_0+\theta\left(1-\frac{1}{\psi}\right)\mu+\theta\kappa_1 A_2\left(1-\nu_1\right)\sigma^2+\frac{1}{2}\left(\theta^2\kappa_1^2 A_2^2 \sigma_\omega^2\right)}{\theta\left(1-\kappa_1\right)}, \tag{C5}$$

where $\theta \equiv \frac{1-\gamma}{1-\frac{1}{\psi}}$, the coefficient of relative risk aversion is γ, the elasticity of intertemporal substitution is ψ, and $0<\beta<1$ is the time discount factor.

Now, we show that the SDF is a function of the innovation in the market variance. To proceed, the log market portfolio return is:

$$\log\left(R_{M,t+1}\right)=\kappa_0+\kappa_1 A_0+\kappa_1 A_1 \rho x_t+\kappa_1 A_1 \varphi_e \sigma_t e_{t+1}+\kappa_1 A_2\left(\sigma^2+\nu_1\left(\sigma_t^2-\sigma^2\right)+\sigma_\omega \omega_{t+1}\right)+g_{t+1}-z_t. \tag{C6}$$

Therefore, the conditional variance of the market portfolio return is:

$$\sigma_{M,t}^2=\left(1+\kappa_1^2 A_1^2 \varphi_e^2\right)\sigma_t^2+\kappa_1^2 A_2^2 \sigma_\omega^2, \tag{C7}$$

which implies

$$\sigma_{M,t+1}^2-E_t\left(\sigma_{M,t+1}^2\right)=\left(1+\kappa_1^2 A_1^2 \varphi_e^2\right)\sigma_\omega \omega_{t+1}. \tag{C8}$$

The expected variance risk premium is

$$E_t\left(\sigma_{M,t+1}^2\right)-E_t^{\mathbb{Q}}\left(\sigma_{M,t+1}^2\right) \;=\; -Cov_t\left[\log\left(\frac{M_{t+1}}{M_t}\right), \sigma_{M,t+1}^2\right] \tag{C9}$$

$$\;=\; \lambda_{m,\omega}\left(1+\kappa_1^2 A_1^2 \varphi_e^2\right)\sigma_\omega^2, \quad \text{with} \quad \lambda_{m,\omega} \equiv (1-\theta)\,\kappa_1 A_2. \tag{C10}$$

The log Epstein and Zin (1991) SDF is

$$\log\left(\frac{M_{t+1}}{M_t}\right) \;=\; \theta\log(\beta)+(\theta-1)\left(\kappa_0-z_t\right)+(\theta-1)\,\kappa_1 z_{t+1}+\left((\theta-1)-\frac{\theta}{\psi}\right)g_{t+1}. \tag{C11}$$

Under our assumptions, we replace g_{t+1} and z_{t+1} in equation (C11) and use (C8) to get

$$\log\left(\frac{M_{t+1}}{M_t}\right) = \zeta_t \ + \ \lambda_{m,\eta}\,\sigma_t\,\eta_{t+1} \ - \ \lambda_{m,e}\,\sigma_t\,e_{t+1} \ - \ \lambda_{m,\omega}\,\sigma_\omega\,\omega_{t+1}, \tag{C12}$$

where

$$\begin{aligned}\zeta_t \ &= \ \theta\log(\beta) + (\theta-1)(\kappa_0 - z_t) + (\theta-1)\,\kappa_1 A_0 + (\theta-1)\,\kappa_1 A_1\,\rho\,x_t \\ &\quad + \left((\theta-1) - \frac{\theta}{\psi}\right)(\mu + x_t) + (\theta-1)\,\kappa_1 A_2\left(\sigma^2 + \nu_1\left(\sigma_t^2 - \sigma^2\right)\right),\end{aligned} \tag{C13}$$

and

$$\lambda_{m,\eta} \equiv (\theta-1) - \frac{\theta}{\psi}, \qquad \lambda_{m,e} \equiv (1-\theta)\,\kappa_1 A_1 \varphi_e, \qquad \lambda_{m,\omega} \equiv (1-\theta)\,\kappa_1 A_2. \tag{C14}$$

We then replace (C8) in (C12):

$$\log\left(\frac{M_{t+1}}{M_t}\right) = \zeta_t + \mathcal{D}_1\left(g_{t+1} - E_t\left(g_{t+1}\right)\right) + \mathcal{D}_2\left(x_{t+1} - E_t\left(x_{t+1}\right)\right) + \mathcal{D}_3\left(\sigma^2_{M,t+1} - E_t\left(\sigma^2_{M,t+1}\right)\right), \tag{C15}$$

where $\mathcal{D}_1$, $\mathcal{D}_2$, and $\mathcal{D}_3$ are

$$\mathcal{D}_1 \ \equiv \ \lambda_{m,\eta}, \qquad \mathcal{D}_2 \ \equiv \ -\lambda_{m,e}/\varphi_e, \qquad \mathcal{D}_3 \ \equiv \ -\frac{\lambda_{m,\omega}}{\left(1 + \kappa_1^2 A_1^2 \varphi_e^2\right)}, \tag{C16}$$

and this was the final step. ■

The permanent and transitory components of the SDF. To derive the permanent and transitory component of SDF, we solve the eigenfunction problem

$$E_t\left(\frac{M_{t+1}}{M_t}\,e\left[\mathcal{X}_{t+1}\right]\right) \ = \ \delta\,e\left[\mathcal{X}_t\right], \tag{C17}$$

where the set of state variables is $\mathcal{X}_t = \begin{pmatrix} x_t \\ \sigma_t^2 - \sigma^2 \end{pmatrix}$. Using the time-series assumptions, it can be shown:

$$\mathcal{X}_{t+1} \ = \ \varphi\,\mathcal{X}_t \ + \ \varphi_{e\omega}\,\xi_{t+1}, \tag{C18}$$

where

$$\varphi = \begin{pmatrix} \rho & 0 \\ 0 & \nu_1 \end{pmatrix}, \varphi_{e\omega} = \begin{pmatrix} \varphi_e \sigma_t & 0 \\ 0 & \sigma_\omega \end{pmatrix}, \text{ and } \xi_{t+1} = \begin{pmatrix} e_{t+1} \\ \omega_{t+1} \end{pmatrix}. \tag{C19}$$

We conjecture that the solution is of the form

$$e[\mathcal{X}_{t+1}] = \exp\left(c^{'} \mathcal{X}_{t+1}\right), \tag{C20}$$

with $c^{'} = (c_1, c_2)$. Proceeding, we denote

$$\Psi_t = E_t\left(\frac{M_{t+1}}{M_t} e[\mathcal{X}_{t+1}]\right) = E_t\left(\exp\left(\log\left(\frac{M_{t+1}}{M_t}\right) + \log\left(e[\mathcal{X}_{t+1}]\right)\right)\right) \tag{C21}$$

and recall that the log SDF is

$$\log\left(\frac{M_{t+1}}{M_t}\right) = \zeta_t + \lambda_{m,\eta}\sigma_t\eta_{t+1} + \Upsilon^{'}\xi_{t+1} \qquad \text{with} \qquad \Upsilon^{'} = \left(-\lambda_{m,e}\sigma_t, -\lambda_{m,\omega}\sigma_\omega\right). \tag{C22}$$

We use (C15) and rewrite (C21) as

$$\Psi_t = \exp\left(\zeta_t + c^{'}\varphi\mathcal{X}_t\right) E_t\left(\exp\left(\lambda_{m,\eta}\sigma_t\eta_{t+1} + \left(\Upsilon^{'} + c^{'}\varphi_{e\omega}\right)\xi_{t+1}\right)\right). \tag{C23}$$

Now,

$$E_t\left(\exp\left(\lambda_{m,\eta}\sigma_t\eta_{t+1} + \left(\Upsilon^{'} + c^{'}\varphi_{e\omega}\right)\xi_{t+1}\right)\right) = \exp\left(\frac{1}{2}\left(\lambda_{m,\eta}^2\sigma_t^2 + \left(\Upsilon^{'}\Upsilon + 2c^{'}\varphi_{e\omega}\Upsilon + c^{'}\varphi_{e\omega}\varphi_{e\omega}^{'}c\right)\right)\right), \tag{C24}$$

and we can, therefore, rewrite (C23) as

$$\Psi_t = \exp\left(\zeta_t + c^{'}\varphi\mathcal{X}_t + \frac{1}{2}\lambda_{m,\eta}^2\sigma_t^2 + \frac{1}{2}\Upsilon^{'}\Upsilon + c^{'}\varphi_{e\omega}\Upsilon + \frac{1}{2}c^{'}\varphi_{e\omega}\varphi_{e\omega}^{'}c\right), \tag{C25}$$

where

$$c^{'}\varphi_{e\omega}\Upsilon = -c_1\lambda_{m,e}\varphi_e\sigma_t^2 - c_2\lambda_{m,\omega}\sigma_\omega^2, \qquad \Upsilon^{'}\Upsilon = \lambda_{m,e}^2\sigma_t^2 + \lambda_{m,\omega}^2\sigma_\omega^2, \tag{C26}$$

$$c^{'}\varphi_{e\omega}\varphi_{e\omega}^{'}c = \varphi_e^2\sigma_t^2c_1^2 + \sigma_\omega^2c_2^2, \qquad c^{'}\varphi\mathcal{X}_t = c_1\rho x_t + \nu_1 c_2\left(\sigma_t^2 - \sigma^2\right). \tag{C27}$$

We recall that

$$\begin{aligned}\zeta_t &= \theta\log(\beta)+(\theta-1)(\kappa_0-A_0)-(\theta-1)A_2\sigma^2+(\theta-1)\kappa_1 A_0+\lambda_{m,\eta}\mu-\lambda_{m,\omega}\sigma^2 \qquad \text{(C28)}\\ &\quad -((\theta-1)A_2+\lambda_{m,\omega}\nu_1)\left(\sigma_t^2-\sigma^2\right)-\left(\frac{\lambda_{m,e}}{\varphi_e}\rho+(\theta-1)A_1-\lambda_{m,\eta}\right)x_t,\end{aligned}$$

and express (C25) as

$$\Psi_t=\delta\exp\left(c'\mathcal{X}_t\right), \qquad \text{(C29)}$$

with

$$\delta=\exp\left(\begin{array}{c}\theta\log(\beta)+(\theta-1)(\kappa_0-A_0)-(\theta-1)A_2\sigma^2+(\theta-1)\kappa_1 A_0\\ +\lambda_{m,\eta}\mu-\lambda_{m,\omega}\sigma^2+\frac{1}{2}\lambda_{m,\eta}^2\sigma^2+\frac{1}{2}\left(\lambda_{m,\omega}^2\sigma_\omega^2\right)+\frac{1}{2}\lambda_{m,e}^2\sigma^2-c_2\lambda_{m,\omega}\sigma_\omega^2\\ -c_1\lambda_{m,e}\varphi_e\sigma^2+\frac{1}{2}\sigma_\omega^2c_2^2+\frac{1}{2}\varphi_e^2c_1^2\sigma^2\end{array}\right),$$

and

$$\begin{aligned}c'\mathcal{X}_t &= c_1\rho x_t+\nu_1c_2\left(\sigma_t^2-\sigma^2\right)+\frac{1}{2}\lambda_{m,\eta}^2\left(\sigma_t^2-\sigma^2\right)+\frac{1}{2}\lambda_{m,e}^2\left(\sigma_t^2-\sigma^2\right) \qquad \text{(C30)}\\ &- c_1\lambda_{m,e}\varphi_e\left(\sigma_t^2-\sigma^2\right)+\frac{1}{2}\varphi_e^2c_1^2\left(\sigma_t^2-\sigma^2\right)-\left(\frac{\lambda_{m,e}}{\varphi_e}\rho+(\theta-1)A_1-\lambda_{m,\eta}\right)x_t. \qquad \text{(C31)}\end{aligned}$$

We use equation (C30) to recover $c'=(c_1,c_2)$ as

$$c_1 = \frac{\lambda_{m,\eta}+(\theta-1)A_1(\kappa_1\rho-1)}{1-\rho}, \qquad c_2 = \frac{\frac{1}{2}\lambda_{m,\eta}^2+\frac{1}{2}\lambda_{m,e}^2-c_1\lambda_{m,e}\varphi_e+\frac{1}{2}\varphi_e^2c_1^2}{1-\nu_1}, \qquad \text{(C32)}$$

and

$$\delta=\exp\left(\begin{array}{c}\theta\log(\beta)+(\theta-1)(\kappa_0-A_0)-(\theta-1)A_2\sigma^2+(\theta-1)\kappa_1A_0\\ +\lambda_{m,\eta}\mu-\lambda_{m,\omega}\sigma^2+c_2(1-\nu_1)\sigma^2+\frac{1}{2}\left(\lambda_{m,\omega}^2\sigma_\omega^2\right)-c_2\lambda_{m,\omega}\sigma_\omega^2+\frac{1}{2}\sigma_\omega^2c_2^2\end{array}\right). \qquad \text{(C33)}$$

The transitory component of the SDF is given by

$$\frac{M_{t+1}^T}{M_t^T}=\delta\exp\left(-c'\left(\mathcal{X}_{t+1}-\mathcal{X}_t\right)\right), \qquad \text{and, hence,} \qquad \frac{M_{t+1}^P}{M_t^P}=\left(\frac{M_{t+1}}{M_t}\right)/\left(\frac{M_{t+1}^T}{M_t^T}\right), \qquad \text{(C34)}$$

yielding (35) of Proposition 4. ■

Appendix D: Solution to the eigenfunction problem in Kelly (2009)

The stochastic discount factor. The dynamics of the real economy are

$$g_{t+1} = \mu + x_t + \sigma_g \sigma_t z_{g,t+1} + \sqrt{\Lambda_t} W_{g,t+1}, \qquad x_{t+1} = \rho_x x_t + \sigma_x \sigma_t z_{x,t+1}, \tag{D1}$$

$$\sigma^2_{t+1} = \overline{\sigma}^2 (1-\rho_\sigma) + \rho_\sigma \sigma_t^2 + \sigma_\sigma z_{\sigma,t+1}, \qquad \Lambda_{t+1} = \overline{\Lambda}(1-\rho_\Lambda) + \rho_\Lambda \Lambda_t + \sigma_\Lambda z_{\Lambda,t+1}. \tag{D2}$$

The z shocks are standard normal and independent. In addition to their gaussian shocks, consumption depends on non-gaussian shocks W_g. The W_g shocks are distributed Laplace with mean zero and variance two, and independent. W_g shock is independent of z shocks. In this model, the SDF is of the form

$$\begin{aligned} \log\left(\frac{M_{t+1}}{M_t}\right) &= \theta \log(\beta) - \frac{\theta}{\psi} g_{t+1} + (\theta - 1)\, r_{M,t+1}, \\ &= \xi_t - \lambda_g \left(\sigma_g \sigma_t z_{g,t+1} + \sqrt{\Lambda_t} W_{g,t+1}\right) - \lambda_x \sigma_x \sigma_t z_{x,t+1} - \lambda_\sigma \sigma_\sigma z_{\sigma,t+1} - \lambda_\Lambda \sigma_\Lambda z_{\Lambda,t+1}, \end{aligned} \tag{D3}$$

with

$$\lambda_g = 1 - \theta + \frac{\theta}{\psi}, \quad \lambda_x = (1-\theta)\kappa_1 A_x, \quad \lambda_\sigma = (1-\theta)\kappa_1 A_\sigma, \quad \lambda_\Lambda = (1-\theta)\kappa_1 A_\Lambda, \quad \text{and,} \tag{D4}$$

$$A_x = \frac{1-\frac{1}{\psi}}{1-\kappa_1\rho_x}, \qquad A_\Lambda = \frac{\theta\left(1-\frac{1}{\psi}\right)\left(1-\frac{1}{\psi}\right)}{1-\kappa_1\rho_x}, \qquad A_\sigma = \frac{\theta}{2}\left(\frac{\left(1-\frac{1}{\psi}\right)^2 \sigma_g^2 + \kappa_1^2 A_x^2 \sigma_x^2}{1-\kappa_1\rho_\sigma}\right), \tag{D5}$$

$$A_0 = \frac{\log(\beta) + \left(1-\frac{1}{\psi}\right)\mu + \kappa_0 + \kappa_1\left(A_\sigma \overline{\sigma}^2 (1-\rho_\sigma) + A_\Lambda \overline{\Lambda}(1-\rho_\Lambda)\right) + \frac{1}{2}\theta\kappa_1^2\left(A_\sigma^2\sigma_\sigma^2 + A_\Lambda^2 \sigma_\Lambda^2\right)}{1-\kappa_1}, \tag{D6}$$

and,

$$\xi_t = \theta \log(\beta) - \frac{\theta}{\psi}(\mu + x_t) + (\theta - 1) E_t\left(r_{M,t+1}\right). \tag{D7}$$

The market return is

$$r_{M,t+1} = \kappa_0 + \kappa_1\, wg_{t+1} - wg_t + g_{t+1}, \tag{D8}$$

with

$$wg_{t+1} = A_0 + A_x x_{t+1} + A_\sigma \sigma^2_{t+1} + A_\Lambda \Lambda_{t+1}. \tag{D9}$$

Therefore,

$$r_{Mt+1} - E_t(r_{Mt+1}) = \kappa_1 A_\Lambda \sigma_\Lambda z_{\Lambda,t+1} + \sigma_g \sigma_t z_{g,t+1} + \sqrt{\Lambda_t} W_{g,t+1} + \kappa_1 A_x \sigma_x \sigma_t z_{x,t+1} + \kappa_1 A_\sigma \sigma_\sigma z_{\sigma,t+1}, \quad \text{(D10)}$$

with

$$E_t(r_{Mt+1}) = \kappa_0 \ + \ \kappa_1 \left(A_0 + A_x E_t x_{t+1} \ + \ A_\sigma E_t(\sigma_{t+1}^2) \ + \ A_\Lambda E_t(\Lambda_{t+1})\right) \ - \ w g_t + E_t(g_{t+1}), \quad \text{(D11)}$$

where

$$E_t(x_{t+1}) \ = \ \rho_x x_t, \qquad E_t(\sigma_{t+1}^2) \ = \ \overline{\sigma}^2 (1 \ - \ \rho_\sigma) \ + \ \rho_\sigma \sigma_t^2, \quad \text{(D12)}$$

$$E_t(\Lambda_{t+1}) \ = \ \overline{\Lambda}(1 \ - \ \rho_\Lambda) \ + \ \rho_\Lambda \Lambda_t, \qquad E_t(g_{t+1}) \ = \ \mu \ + \ x_t. \quad \text{(D13)}$$

The next step is to show that the SDF depends on the market variance. In this regard,

$$Var_t\left(r_{Mt+1}\right) = \kappa_1^2 \left(A_\Lambda^2 \sigma_\Lambda^2 \ + \ A_\sigma^2 \sigma_\sigma^2\right) \ + \ \left(\sigma_g^2 \ + \ \kappa_1^2 A_x^2 \sigma_x^2\right) \sigma_t^2 \ + \ Var_t\left(W_{g,t+1}\right) \Lambda_t. \quad \text{(D14)}$$

Now the moment generating function of the Laplace variable W_g is

$$E_t\left(\exp\left(s W_{g,t+1}\right)\right) = \frac{1}{1-s^2}. \quad \text{(D15)}$$

Therefore, $Var_t\left(W_{g,t+1}\right) = 2$, and the conditional variance of the market return is

$$\sigma_{Mt+1}^2 \ = \ \kappa_1^2 \left(A_\Lambda^2 \sigma_\Lambda^2 \ + \ A_\sigma^2 \sigma_\sigma^2\right) \ + \ \left(\sigma_g^2 \ + \ \kappa_1^2 A_x^2 \sigma_x^2\right) \sigma_{t+1}^2 \ + \ 2\Lambda_{t+1}, \quad \text{(D16)}$$

and the innovation in the market variance is

$$\sigma_{Mt+1}^2 - E_t\left(\sigma_{Mt+1}^2\right) \ = \ \left(\sigma_g^2 + \kappa_1^2 A_x^2 \sigma_x^2\right) \sigma_\sigma z_{\sigma,t+1} \ + \ 2\sigma_\Lambda z_{\Lambda,t+1}. \quad \text{(D17)}$$

The expected variance risk premium is

$$E_t\left(\sigma_{Mt+1}^2\right) - E_t^{\mathbb{Q}}\left(\sigma_{Mt+1}^2\right) \ = \ -Cov_t\left[\log\left(\frac{M_{t+1}}{M_t}\right), \sigma_{Mt+1}^2 - E_t\left(\sigma_{Mt+1}^2\right)\right], \quad \text{(D18)}$$

$$= \ \lambda_\sigma\left(\sigma_g^2 + \kappa_1^2 A_x^2 \sigma_x^2\right) \sigma_\sigma^2 + 2\lambda_\Lambda \sigma_\Lambda^2. \quad \text{(D19)}$$

We replace the conditional market variance in the SDF and get

$$\begin{aligned}\log\left(\frac{M_{t+1}}{M_t}\right) &= \xi_t + \mathcal{D}_1\left(g_{t+1}-E_t\left(g_{t+1}\right)\right) + \mathcal{D}_2\left(x_{t+1}-E_t\left(x_{t+1}\right)\right)\\ &\quad +\mathcal{D}_3\left(\sigma^2_{Mt+1}-E_t\left(\sigma^2_{Mt+1}\right)\right) + \mathcal{D}_4\left(\Lambda_{t+1}-E_t\left(\Lambda_{t+1}\right)\right),\end{aligned}$$

with

$$\mathcal{D}_1=-\lambda_g,\qquad \mathcal{D}_2=-\lambda_x,\qquad \mathcal{D}_3=-\frac{\lambda_\sigma}{\left(\sigma_g^2+\kappa_1^2A_x^2\sigma_x^2\right)},\qquad \mathcal{D}_4=\frac{2\lambda_\sigma}{\left(\sigma_g^2+\kappa_1^2A_x^2\sigma_x^2\right)}-\lambda_\Lambda. \tag{D20}$$

The permanent and transitory component of the SDF. We consider the set of state variables

$$\mathcal{Z}_t=\begin{pmatrix} x_t\\ \sigma_t^2-\overline{\sigma}^2\\ \Lambda_t-\overline{\Lambda}\end{pmatrix}. \tag{D21}$$

The goal is to solve the eigenfunction problem $E_t\left(\frac{M_{t+1}}{M_t}e\left[\mathcal{Z}_{t+1}\right]\right)=\delta e\left[\mathcal{Z}_t\right]$. Using our time series assumptions,

$$\mathcal{Z}_{t+1}=\Gamma\mathcal{Z}_t+\Sigma\eta_{t+1}, \tag{D22}$$

with

$$\Gamma=\begin{pmatrix}\rho_x & 0 & 0\\ 0 & \rho_\sigma & 0\\ 0 & 0 & \rho_\Lambda\end{pmatrix},\quad \Sigma=\begin{pmatrix}\sigma_x\sigma_t & 0 & 0\\ 0 & \sigma_\sigma & 0\\ 0 & 0 & \sigma_\Lambda\end{pmatrix},\quad \eta_{t+1}=\begin{pmatrix} z_{x,t+1}\\ z_{\sigma,t+1}\\ z_{\Lambda,t+1}\end{pmatrix}. \tag{D23}$$

We conjecture that the solution is of the form

$$e\left[\mathcal{Z}_{t+1}\right] = \exp\left(c'\mathcal{Z}_{t+1}\right), \tag{D24}$$

with $c=(c_1,c_2,c_3)$. Now,

$$\begin{aligned}\Psi &= E_t\left(\frac{M_{t+1}}{M_t}e\left[\mathcal{Z}_{t+1}\right]\right)=E_t\left(\exp\left(\log\left(\frac{M_{t+1}}{M_t}\right) + \log\left(e\left[\mathcal{Z}_{t+1}\right]\right)\right)\right)\\ &= E_t\left(\exp\left(\xi_t+c'\Gamma\mathcal{Z}_t-\lambda_g\sigma_g\sigma_t z_{g,t+1}-\left(\varphi'-c'\Sigma\right)\eta_{t+1}\right)\exp\left(\lambda_g\sqrt{\Lambda_t}W_{g,t+1}\right)\right).\end{aligned} \tag{D25}$$

Since $W_{g,t+1}$ is independent of the z variables, it follows that

$$\Psi \;=\; \exp\left(\xi_t + c^{'}\Gamma \mathcal{Z}_t\right)\Upsilon_1\Upsilon_2, \tag{D26}$$

with $\varphi^{'} = (\lambda_x\sigma_x\sigma_t, \lambda_\sigma\sigma_\sigma, \lambda_\Lambda\sigma_\Lambda)$, and

$$\Upsilon_1 = E_t\left(\exp\left(-\lambda_g\sigma_g\sigma_t z_{g,t+1} - \left(\varphi^{'} - c^{'}\Sigma\right)\eta_{t+1}\right)\right), \quad \Upsilon_2 = E_t\left(\exp\left(-\lambda_g\sqrt{\Lambda_t}W_{g,t+1}\right)\right). \tag{D27}$$

We notice that

$$\Upsilon_1 = \exp\left(\frac{1}{2}\left(\lambda_g^2\sigma_g^2\sigma_t^2\right) + \frac{1}{2}\left(\varphi^{'} - c^{'}\Sigma\right)(\varphi - \Sigma c)\right), \quad \Upsilon_2 = E_t\left(\exp\left(-\lambda_g\sqrt{\Lambda_t}W_{g,t+1}\right)\right) = \frac{1}{1-\lambda_g^2\Lambda_t} \tag{D28}$$

and, hence, rewrite (D26) as

$$\Psi = \exp\left(\xi_t + c^{'}\Gamma \mathcal{Z}_t\right)\exp\left(\frac{1}{2}\left(\lambda_g^2\sigma_g^2\sigma_t^2\right) + \frac{1}{2}\left(\varphi^{'} - c^{'}\Sigma\right)(\varphi - \Sigma c)\right)\exp\left(\log\left(\frac{1}{1-\lambda_g^2\Lambda_t}\right)\right). \tag{D29}$$

Following Kelly (2009), we use the Taylor expansion of $\frac{1}{1-\lambda_g^2\Lambda_t}$ to get $\frac{1}{1-\lambda_g^2\Lambda_t} \approx \lambda_g^2\Lambda_t$. Therefore, (D29) simplifies to

$$\begin{aligned}
\Psi &= \exp\left(\xi_t + c^{'}\Gamma \mathcal{Z}_t\right)\exp\left(\frac{1}{2}\left(\lambda_g^2\sigma_g^2\sigma_t^2\right) + \frac{1}{2}\left(\varphi^{'} - c^{'}\Sigma\right)(\varphi - \Sigma c)\right)\exp\left(\lambda_g^2\Lambda_t\right), \\
&= \exp\left(\xi_t + c^{'}\Gamma \mathcal{Z}_t + \lambda_g^2\Lambda_t + \frac{1}{2}\lambda_g^2\sigma_g^2\sigma_t^2 + \frac{1}{2}\left(\varphi^{'} - c^{'}\Sigma\right)(\varphi - \Sigma c)\right), \\
&= \exp\left(\xi_t + c^{'}\Gamma \mathcal{Z}_t + \lambda_g^2\Lambda_t + \frac{1}{2}\lambda_g^2\sigma_g^2\sigma_t^2 + \frac{1}{2}\varphi^{'}\varphi - \varphi^{'}\Sigma c + \frac{1}{2}c^{'}\Sigma\Sigma c\right).
\end{aligned} \tag{D30}$$

This implies that

$$c^{'}\Gamma \mathcal{Z}_t = c_1\rho_x x_t + c_2\rho_\sigma\left(\sigma_t^2 - \overline{\sigma}^2\right) + c_3\rho_\Lambda\left(\Lambda_t - \overline{\Lambda}\right), \qquad \varphi^{'}\varphi = \lambda_x^2\sigma_x^2\sigma_t^2 + \lambda_\sigma^2\sigma_\sigma^2 + \lambda_\Lambda^2\sigma_\Lambda^2,$$

$$\varphi^{'}\Sigma c = \lambda_x\sigma_x^2\sigma_t^2 c_1 + \lambda_\sigma\sigma_\sigma^2 c_2 + \lambda_\Lambda\sigma_\Lambda^2 c_3, \qquad c^{'}\Sigma\Sigma c = c_1^2\sigma_x^2\sigma_t^2 + c_2^2\sigma_\sigma^2 + c_3^2\sigma_\Lambda^2,$$

and we rewrite (D30) as

$$\Psi = \exp\begin{pmatrix} \xi_t + c'\Gamma Z_t + \lambda_g^2\Lambda_t + \frac{1}{2}\lambda_g^2\sigma_g^2\sigma_t^2 + \frac{1}{2}\left(\lambda_x^2\sigma_x^2\sigma_t^2 + \lambda_\sigma^2\sigma_\sigma^2 + \lambda_\Lambda^2\sigma_\Lambda^2\right) \\ -\left(\lambda_x\sigma_x^2\sigma_t^2 c_1 + \lambda_\sigma\sigma_\sigma^2 c_2 + \lambda_\Lambda\sigma_\Lambda^2 c_3\right) + \frac{1}{2}\left(c_1^2\sigma_x^2\sigma_t^2 + c_2^2\sigma_\sigma^2 + c_3^2\sigma_\Lambda^2\right) \end{pmatrix}, \tag{D31}$$

where

$$\begin{aligned} \xi_t = {} & \theta\log(\beta) + (\theta-1)\kappa_0 + (\theta-1)\kappa_1 A_0 - \frac{\theta}{\psi}\mu + (\theta-1)\kappa_1 A_\sigma\overline{\sigma}^2(1-\rho_\sigma) \\ & + (\theta-1)\mu + (\theta-1)\kappa_1 A_\Lambda\overline{\Lambda}(1-\rho_\Lambda) - (\theta-1)\left(A_0 + A_\sigma\overline{\sigma}^2 + A_\Lambda\overline{\Lambda}\right) \\ & + (\theta-1)A_\Lambda\kappa_1\rho_\Lambda\overline{\Lambda} + (\theta-1)A_\sigma\kappa_1\rho_\sigma\overline{\sigma}^2\left(-\frac{\theta}{\psi} + (\theta-1)\kappa_1 A_x\rho_x + (\theta-1) - (\theta-1)A_x\right)x_t \\ & + (\theta-1)A_\sigma(\kappa_1\rho_\sigma - 1)\left(\sigma_t^2 - \overline{\sigma}^2\right) + (\theta-1)A_\Lambda(\kappa_1\rho_\Lambda - 1)\left(\Lambda_t - \overline{\Lambda}\right). \end{aligned} \tag{D32}$$

We can, therefore, express (D31) as $\Psi = \delta\exp\left(c'Z_t\right)$, with

$$\delta = \exp\begin{pmatrix} \theta\log(\beta) + (\theta-1)\kappa_0 + (\theta-1)\kappa_1 A_0 - \frac{\theta}{\psi}\mu + (\theta-1)\kappa_1 A_\sigma\overline{\sigma}^2(1-\rho_\sigma) \\ +(\theta-1)\mu + (\theta-1)\kappa_1 A_\Lambda\overline{\Lambda}(1-\rho_\Lambda) - (\theta-1)\left(A_0 + A_\sigma\overline{\sigma}^2 + A_\Lambda\overline{\Lambda}\right) \\ +(\theta-1)A_\Lambda\kappa_1\rho_\Lambda\overline{\Lambda} + (\theta-1)A_\sigma\kappa_1\rho_\sigma\overline{\sigma}^2 \\ +\frac{1}{2}\left(\lambda_\sigma^2\sigma_\sigma^2 + \lambda_\Lambda^2\sigma_\Lambda^2\right) - \left(\lambda_\sigma\sigma_\sigma^2 c_2 + \lambda_\Lambda\sigma_\Lambda^2 c_3\right) + \frac{1}{2}\left(c_2^2\sigma_\sigma^2 + c_3^2\sigma_\Lambda^2\right) + \lambda_g^2\overline{\Lambda} \\ +\left(-\lambda_x\sigma_x^2 c_1 + \frac{1}{2}c_1^2\sigma_x^2 + \frac{1}{2}\lambda_g^2\sigma_g^2 + \frac{1}{2}\lambda_x^2\sigma_x^2\right)\overline{\sigma}^2 \end{pmatrix} \tag{D33}$$

$$c = \begin{pmatrix} c_1\rho_x + (-\lambda_g + (\theta-1)A_x(\kappa_1\rho_x - 1)) \\ c_2\rho_\sigma - \lambda_x\sigma_x^2 c_1 + \frac{1}{2}c_1^2\sigma_x^2 + \frac{1}{2}\lambda_g^2\sigma_g^2 + \frac{1}{2}\lambda_x^2\sigma_x^2 + (\theta-1)A_\sigma(\kappa_1\rho_\sigma - 1) \\ c_3\rho_\Lambda + \lambda_g^2 + (\theta-1)A_\Lambda(\kappa_1\rho_\Lambda - 1) \end{pmatrix}. \tag{D34}$$

Solving (D34) for the individual components in $c = (c_1, c_2, c_3)$ allows us to deduce that

$$c_1 = \frac{-\lambda_g + (\theta-1)A_x(\kappa_1\rho_x - 1)}{1-\rho_x}, \tag{D35}$$

$$c_2 = \frac{-\lambda_x\sigma_x^2 c_1 + \frac{1}{2}c_1^2\sigma_x^2 + \frac{1}{2}\lambda_g^2\sigma_g^2 + \frac{1}{2}\lambda_x^2\sigma_x^2 + (\theta-1)A_\sigma(\kappa_1\rho_\sigma - 1)}{1-\rho_\sigma}, \tag{D36}$$

$$c_3 = \frac{\lambda_g^2 + (\theta-1)A_\Lambda(\kappa_1\rho_\Lambda - 1)}{1-\rho_\Lambda}. \tag{D37}$$

The transitory component of the SDF is

$$\frac{M_{t+1}^T}{M_t^T} = \delta \exp\left(-c^{'}\left(\mathcal{Z}_{t+1} - \mathcal{Z}_t\right)\right), \tag{D38}$$

and the permanent component is determined accordingly. ■

Appendix E: Restrictions implied by an asset pricing model that explains the equity premium and yet fails to satisfy the lower bound on the variance of the permanent component of the SDF

The essence of the result is captured in the following Lemma. The converse result on the restriction implied by an asset pricing model that satisfies the lower bound on the variance of the permanent component of SDFs and fails to explain the equity premium puzzle is available from the authors.

Lemma 2 *Suppose the SDF implied from an asset pricing model can be decomposed into a permanent component and a transitory component. Let* $\mu_{R_\infty} \equiv E\left(\frac{1}{R_{t+1,\infty}^2}\right)$ *and, as before,* $\mu_m \equiv E\left(\frac{M_{t+1}}{M_t}\right)$*, and suppose that the asset pricing model satisfies two restrictions:*

$$(a)\; Var\left[\frac{M_{t+1}}{M_t}\right] \geq \sigma_{HJ}^2, \text{ where } \sigma_{HJ}^2 \text{ is defined in Hansen and Jagannathan } (1991,\text{ Eq. } (12)), \text{ and} \tag{E1}$$

$$(b)\; Cov\left[\left(\frac{M_{t+1}^p}{M_t^p}\right)^2, \left(\frac{M_{t+1}^T}{M_t^T}\right)^2\right] \leq \Delta_c \;\equiv\; \mu_m^2\left(1+\frac{\sigma_{HJ}^2}{\mu_m^2}\right) \;-\; \mu_{R_\infty}\left(1+\sigma_{pc}^2\right). \tag{E2}$$

If (a) and (b) hold, then $Var\left[\frac{M_{t+1}^P}{M_t^P}\right] \geq \sigma_{pc}^2$*, where* σ_{pc}^2 *is defined in (8) of Proposition 1. Alternatively, if (a) and* $Var\left[\frac{M_{t+1}^P}{M_t^P}\right] \leq \sigma_{pc}^2$ *hold, then* $Cov\left[\left(\frac{M_{t+1}^p}{M_t^p}\right)^2, \left(\frac{M_{t+1}^T}{M_t^T}\right)^2\right] \geq \Delta_c$.

Proof: If an asset pricing model satisfies the Hansen and Jagannathan (1991) variance bound (i.e., restriction (a)), it amounts to a resolution of the equity premium puzzle. Proceeding,

$$Var\left[\frac{M_{t+1}}{M_t}\right] = E\left(\left(\frac{M_{t+1}^p}{M_t^p}\right)^2\left(\frac{M_{t+1}^T}{M_t^T}\right)^2\right) - \mu_m^2, \tag{E3}$$

$$= Cov\left[\left(\frac{M_{t+1}^p}{M_t^p}\right)^2, \left(\frac{M_{t+1}^T}{M_t^T}\right)^2\right] + \left(E\left(\frac{M_{t+1}^T}{M_t^T}\right)^2\right)\left(1+Var\left[\frac{M_{t+1}^p}{M_t^p}\right]\right) - \mu_m^2. \tag{E4}$$

Consider the SDF $\frac{M_{t+1}^*}{M_t^*}$ that displays the minimum variance property, namely, $Var\left[\frac{M_{t+1}^*}{M_t^*}\right] = \sigma_{HJ}^2$ (Hansen and Jagannathan (1991)). Consider also the permanent component of the SDF $\frac{M_{t+1}^{*P}}{M_t^{*P}}$ that displays the minimum variance property, namely, $Var\left[\frac{M^{*P}_{t+1}}{M^{*P}_t}\right] = \sigma^2_{pc}$ (see equation (28)). The SDF with the minimum variance property admits a unique decomposition (e.g., Roman (2007, Theorem 9.15, page 220))

$$\frac{M^*_{t+1}}{M^*_t} \; = \; \text{proj}\left[\frac{M^*_{t+1}}{M^*_t}\middle|\frac{M^{*P}_{t+1}}{M^{*P}_t}\right] \; + \; \varepsilon_{t+1}, \qquad \text{with} \qquad E\left(\varepsilon_{t+1}\frac{M^{*P}_{t+1}}{M^{*P}_t}\right) = 0, \tag{E5}$$

where $\text{proj}\left[\frac{M^*_{t+1}}{M^*_t}\middle|\frac{M^{*P}_{t+1}}{M^{*P}_t}\right]$ represents the projection of $\frac{M^*_{t+1}}{M^*_t}$ onto $\frac{M^{*P}_{t+1}}{M^{*P}_t}$, i.e., $\text{proj}\left[\frac{M^*_{t+1}}{M^*_t}\middle|\frac{M^{*P}_{t+1}}{M^{*P}_t}\right] = \frac{E\left(\frac{M^*_{t+1}}{M^*_t}\frac{M^{*P}_{t+1}}{M^{*P}_t}\right)}{E\left(\frac{M^{*P}_{t+1}}{M^{*P}_t}\right)^2}\frac{M^{*P}_{t+1}}{M^{*P}_t}$. For future use, define

$$\overline{b} \; \equiv \; \frac{E\left(\frac{M^*_{t+1}}{M^*_t}\frac{M^{*P}_{t+1}}{M^{*P}_t}\right)}{E\left(\frac{M^{*P}_{t+1}}{M^{*P}_t}\right)^2}, \qquad \text{hence,} \qquad \frac{M^*_{t+1}}{M^*_t} \; = \; \overline{b}\left(\frac{M^{*P}_{t+1}}{M^{*P}_t}\right) \; + \; \varepsilon_{t+1}. \tag{E6}$$

Therefore,

$$E\left(\left(\frac{M^*_{t+1}}{M^*_t}\right)^2\right) \; = \; \overline{b}^2 E\left(\left(\frac{M^{*P}_{t+1}}{M^{*P}_t}\right)^2\right) \; + \; E(\varepsilon^2_{t+1}). \tag{E7}$$

Rearranging,

$$\sigma^2_{HJ} \; = \; \overline{b}^2\left(\sigma^2_{pc}+1\right) \; + \; E(\varepsilon^2_{t+1}) \; - \; \mu^2_m. \tag{E8}$$

Impose the condition that the asset pricing model explains the equity premium puzzle. Then,

$$Var\left[\frac{M_{t+1}}{M_t}\right] \; \geq \; \sigma^2_{HJ} \; = \; \underbrace{\overline{b}^2\left(\sigma^2_{pc}+1\right) \; + \; E(\varepsilon^2_{t+1}) \; - \; \mu^2_m}_{\text{from equation (E8)}}. \tag{E9}$$

Combining (E4) and (E9), we arrive at

$$Cov\left[\left(\frac{M^p_{t+1}}{M^p_t}\right)^2, \left(\frac{M^T_{t+1}}{M^T_t}\right)^2\right] + \left(E\left(\frac{M^T_{t+1}}{M^T_t}\right)^2\right)\left(1+Var\left[\frac{M^p_{t+1}}{M^p_t}\right]\right) - \mu^2_m \geq \overline{b}^2\left(\sigma^2_{pc}+1\right) + E\left(\varepsilon^2_{t+1}\right) - \mu^2_m. \tag{E10}$$

Rearranging and using equation (E8),

$$Var\left[\frac{M^p_{t+1}}{M^p_t}\right] \; \geq \; \Delta_d \; + \; \sigma^2_{pc}, \tag{E11}$$

where recognizing that $\left(\frac{M_{t+1}^T}{M_t^T}\right)^2 = R_{t+1,\infty}^{-2}$ and letting

$$\Delta_d \equiv \left(E\left(1/R_{t+1,\infty}^2\right)\right)^{-1}\left(\mu_m^2+\sigma_{HJ}^2\right)-\left(1+\sigma_{pc}^2\right)-\left(E\left(1/R_{t+1,\infty}^2\right)\right)^{-1} Cov\left[\left(\frac{M_{t+1}^p}{M_t^p}\right)^2,\left(\frac{M_{t+1}^T}{M_t^T}\right)^2\right]. \quad \text{(E12)}$$

Based on (E11), we may deduce the following:

- If $Var\left[\frac{M_{t+1}^p}{M_t^p}\right] \leq \sigma_{pc}^2$, then $\Delta_d \leq 0$;
- Alternatively, if $\Delta_d \geq 0$, then $Var\left[\frac{M_{t+1}^p}{M_t^p}\right] \geq \sigma_{pc}^2$.

To derive the condition under which $\Delta_d \geq 0$, observe that

$$\Delta_d \geq 0 \Leftrightarrow \Delta_c \geq Cov\left[\left(\frac{M_{t+1}^p}{M_t^p}\right)^2,\left(\frac{M_{t+1}^T}{M_t^T}\right)^2\right], \quad \text{(E13)}$$

where Δ_c is as defined in (E2), delivering the statement of the Lemma. Δ_c is computable from the returns data. ■

Table 1
Variance of the permanent and transitory component of the SDFs from long-run risk models

Parameters in Table Appendix-I are employed to compute the variance measure for the permanent component, transitory component, and the ratio of the permanent to the transitory component for the asset pricing models of Bansal and Yaron (2004) (based on equation (35)) and Kelly (2009) (based on equation (42)). Reported values are averaged over 10,000 replications. Each replication is obtained by simulating the models based on the shocks in (34) and (41), respectively, while keeping fixed the assumed parameters in Table Appendix-I. SET 1 contains the risk-free bond, the long-term bond, the market, and the 25 Fama-French equity portfolios sorted by size and book-to-market. SET 2 contains the risk-free bond, the long-term bond, the market, and the 25 Fama-French equity portfolios sorted by size and momentum. The monthly data used in the construction of the bounds is from 1931:07 to 2009:12 (942 observations). The data on the risk-free bond and equity are from the data library of Kenneth French. The data on long-term bonds is from Morningstar. Reported are the estimates from the data, along with the 90% confidence intervals in square brackets. To obtain the confidence intervals, we create 10,000 random samples of size 942 from the data, where the sampling in the block bootstrap is based on 15 blocks. The reported bound on $Var\left[\frac{M_{t+1}^P}{M_t^P}/\frac{M_{t+1}^T}{M_t^T}\right]$ corresponds to $\mu_{m^P/m^T} = 0.995$ (the mean of the ratio of the permanent component of the SDF to the transitory component of the SDF), close to that obtained with both asset pricing models. The p-values, shown in curly brackets, represent the number of times the model-based variance measure exceeds the corresponding bound from the data, as in SET 1.

	Panel A: Asset pricing models		*Panel B: Bounds from returns data*	
	Bansal and Yaron	Kelly	SET 1	SET 2
$Var\left[\frac{M_{t+1}^P}{M_t^P}\right]$	0.034	0.084	0.105	0.124
(monthly)	{0.000}	{0.038}	[0.029,0.108]	[0.034,0.132]
$Var\left[\frac{M_{t+1}^P}{M_t^P}\right]/Var\left[\frac{M_{t+1}}{M_t}\right]$	1.260	1.095	0.972	0.976
	{0.000}	{0.000}	[0.961,0.973]	[0.966,0.978]
$Var\left[\frac{M_{t+1}^T}{M_t^T}\right]/Var\left[\frac{M_{t+1}}{M_t}\right]$	0.071	0.015	0.006	0.005
	{0.000}	{0.000}	[0.005,0.008]	[0.004,0.007]
$Var\left[\frac{M_{t+1}^P}{M_t^P}/\frac{M_{t+1}^T}{M_t^T}\right]$	0.045	0.094	0.138	0.150
(monthly, $\mu_{m^P/m^T} = 0.995$)	{0.000}	{0.000}	[0.052,0.145]	[0.052,0.162]
Equity premium	5.534	6.046		
(annualized, %)	[4.641,6.428]	[5.766, 6.327]		
Risk-free return	2.578	3.111		
(annualized, %)	[1.497,3.660]	[2.082,4.141]		

Table 2

Bounds, equity premium, and risk-free return implied by the Bansal and Yaron (2004) model under alternative parameterizations of preferences

We fix the consumption growth process parameters as implied by the nondurable consumption data (see Table Appendix-I), but vary (i) the subjective discount factor β, (ii) the risk aversion γ, and (iii) the elasticity of intertemporal substitution ψ. In addition to reporting the variance bounds on the permanent and the transitory component of the SDF for the Bansal and Yaron (2004) model, we also report the mean (in %, annualized) and the standard deviation (in %, annualized) of the model implied equity risk premium and the risk-free return. Reported values are averaged over 10,000 replications. Each replication is obtained by simulating the model based on the shocks in (34), while keeping fixed the consumption growth parameters.

	β	γ	ψ	$Var\left[\frac{M^P_{t+1}}{M^P_t}\right]$	$\frac{Var\left[\frac{M^P_{t+1}}{M^P_t}\right]}{Var\left[\frac{M_{t+1}}{M_t}\right]}$	$\frac{Var\left[\frac{M^T_{t+1}}{M^T_t}\right]}{Var\left[\frac{M_{t+1}}{M_t}\right]}$	$Var\left[\frac{\frac{M^P_{t+1}}{M^P_t}}{\frac{M^T_{t+1}}{M^T_t}}\right]$	μ_{m^P/m^T}	Model Properties (%)			
									Equity Premium		Risk-free Return	
									Mean	Std.	Mean	Std.
L/L/L	0.997	5	0.5	0.008	2.216	0.341	0.015	1.007	0.163	14.46	7.109	1.05
L/L/M	0.997	5	1.5	0.008	1.319	0.041	0.010	1.004	1.481	19.05	4.333	0.35
L/L/H	0.997	5	2.0	0.008	1.241	0.030	0.009	1.003	1.679	19.72	3.943	0.26
L/M/L	0.997	7.5	0.5	0.018	1.769	0.163	0.030	1.008	1.197	14.48	7.190	1.04
L/M/M	0.997	7.5	1.5	0.018	1.264	0.049	0.023	1.002	3.450	19.13	4.053	0.35
L/M/H	0.997	7.5	2.0	0.018	1.217	0.043	0.022	1.002	3.759	19.81	3.636	0.27
L/H/L	0.997	10	0.5	0.034	1.654	0.140	0.055	1.012	2.293	14.55	7.290	1.04
L/H/M	0.997	10	1.5	0.034	1.263	0.072	0.045	0.993	5.529	19.29	3.791	0.35
L/H/H	0.997	10	2.0	0.034	1.236	0.067	0.045	0.996	5.967	19.98	3.313	0.27
M/L/L	0.998	5	0.5	0.008	2.212	0.341	0.015	1.006	0.163	14.46	5.911	1.05
M/L/M	0.998	5	1.5	0.008	1.316	0.041	0.010	1.003	1.482	19.05	3.126	0.35
M/L/H	0.998	5	2.0	0.008	1.239	0.030	0.009	1.002	1.681	19.72	2.739	0.26
M/M/L	0.998	7.5	0.5	0.018	1.766	0.163	0.030	1.007	1.198	14.48	6.010	1.05
M/M/M	0.998	7.5	1.5	0.018	1.262	0.049	0.023	1.001	3.445	19.13	2.854	0.35
M/M/H	0.998	7.5	2.0	0.018	1.214	0.043	0.022	1.001	3.760	19.81	2.433	0.27
M/H/L	0.998	10	0.5	0.034	1.650	0.140	0.055	1.011	2.297	14.55	6.103	1.05
M/H/M	0.998	10	1.5	0.034	1.260	0.071	0.045	0.992	5.534	19.29	2.579	0.35
M/H/H	0.998	10	2.0	0.034	1.234	0.067	0.044	0.995	5.957	19.98	2.108	0.27
H/L/L	0.999	5	0.5	0.008	2.207	0.341	0.015	1.005	0.163	14.46	4.739	1.04
H/L/M	0.999	5	1.5	0.008	1.313	0.041	0.010	1.002	1.481	19.05	1.915	0.35
H/L/H	0.999	5	2.0	0.008	1.236	0.030	0.009	1.001	1.678	19.72	1.543	0.26
H/M/L	0.999	7.5	0.5	0.018	1.763	0.163	0.030	1.006	1.198	14.48	4.824	1.05
H/M/M	0.999	7.5	1.5	0.018	1.259	0.049	0.023	1.000	3.446	19.13	1.659	0.35
H/M/H	0.999	7.5	2.0	0.018	1.212	0.043	0.022	1.000	3.762	19.81	1.231	0.27
H/H/L	0.999	10	0.5	0.034	1.648	0.141	0.055	1.010	2.296	14.55	4.890	1.04
H/H/M	0.999	10	1.5	0.034	1.258	0.072	0.045	0.991	5.524	19.29	1.385	0.35
H/H/H	0.999	10	2.0	0.034	1.231	0.067	0.044	0.994	5.965	19.98	0.896	0.27

Table Appendix-I
Parameterizations of asset pricing models in the long-run risk class

The long-run risk model of Bansal and Yaron (2004) is based on

$$g_{t+1} = \mu + x_t + \sigma_t \eta_{t+1}, \qquad x_{t+1} = \rho x_t + \varphi_e \sigma_t e_{t+1},$$
$$\sigma^2_{t+1} = \sigma^2 + \nu_1 \left(\sigma^2_t - \sigma^2\right) + \sigma_\omega \omega_{t+1}, \qquad \omega_{t+1},\, e_{t+1},\, \eta_{t+1} \sim i.i.d\ \mathcal{N}(0,1),$$

and the long-run risk model of Kelly (2009) is based on

$$g_{t+1} = \mu + x_t + \sigma_g \sigma_t z_{g,t+1} + \sqrt{\Lambda_t} W_{g,t+1}, \qquad x_{t+1} = \rho_x x_t + \sigma_x \sigma_t z_{x,t+1},$$
$$\sigma^2_{t+1} = \overline{\sigma}^2 (1-\rho_\sigma) + \rho_\sigma \sigma^2_t + \sigma_\sigma z_{\sigma,t+1}, \qquad \Lambda_{t+1} = \overline{\Lambda}(1-\rho_\Lambda) + \rho_\Lambda \Lambda_t + \sigma_\Lambda z_{\Lambda,t+1},$$
$$z_{g,t+1},\, z_{x,t+1},\, z_{\sigma,t+1},\, z_{\Lambda,t+1} \sim i.i.d\ \mathcal{N}(0,1), \qquad W_{g,t+1} \sim \text{Laplace}(0,1).$$

Parameters chosen for the Epstein and Zin (1991) utility function, the mean consumption growth (μ), the predictable consumption component (ρ and φ_e), and the conditional volatility of the log consumption growth (σ, ν_1, σ_ω) are in line with the studies of Bansal and Yaron (2004), Bansal, Kiku, and Yaron (2009), Beeler and Campbell (2009), Constantinides and Ghosh (2008), Koijen, Lustig, Van Nieuwerburgh, and Verdelhan (2010), and Yang (2010). β is the time preference parameter, γ is the risk aversion parameter, and ψ is the parameter of the elasticity of intertemporal substitution. Parameters of the conditional volatility of the tails are chosen to plausibly mimic the process that governs the conditional volatility of the consumption growth in Bansal and Yaron (2004). The consumption growth parameters correspond to U.S. nondurable consumption data.

Panel A: Bansal and Yaron (2004)

Consumption growth

μ	ρ	φ_e	σ	ν_1	σ_ω
0.0015	0.979	0.044	0.0078	0.987	0.23×10^{-5}

Preferences

β	γ	ψ
0.998	10	1.5

Panel B: Kelly (2009)

Consumption growth

μ	σ_g	ρ_x	σ_x	σ	ρ_σ	σ_σ
0.0015	0.090	0.720	0.090	0.0078	0.978	0.001

Preferences

β	γ	ψ
0.980	10	1.5

Tail risk

$\overline{\Lambda}$	ρ_Λ	σ_Λ
$(0.015)^2$	0.978	0.17×10^{-5}

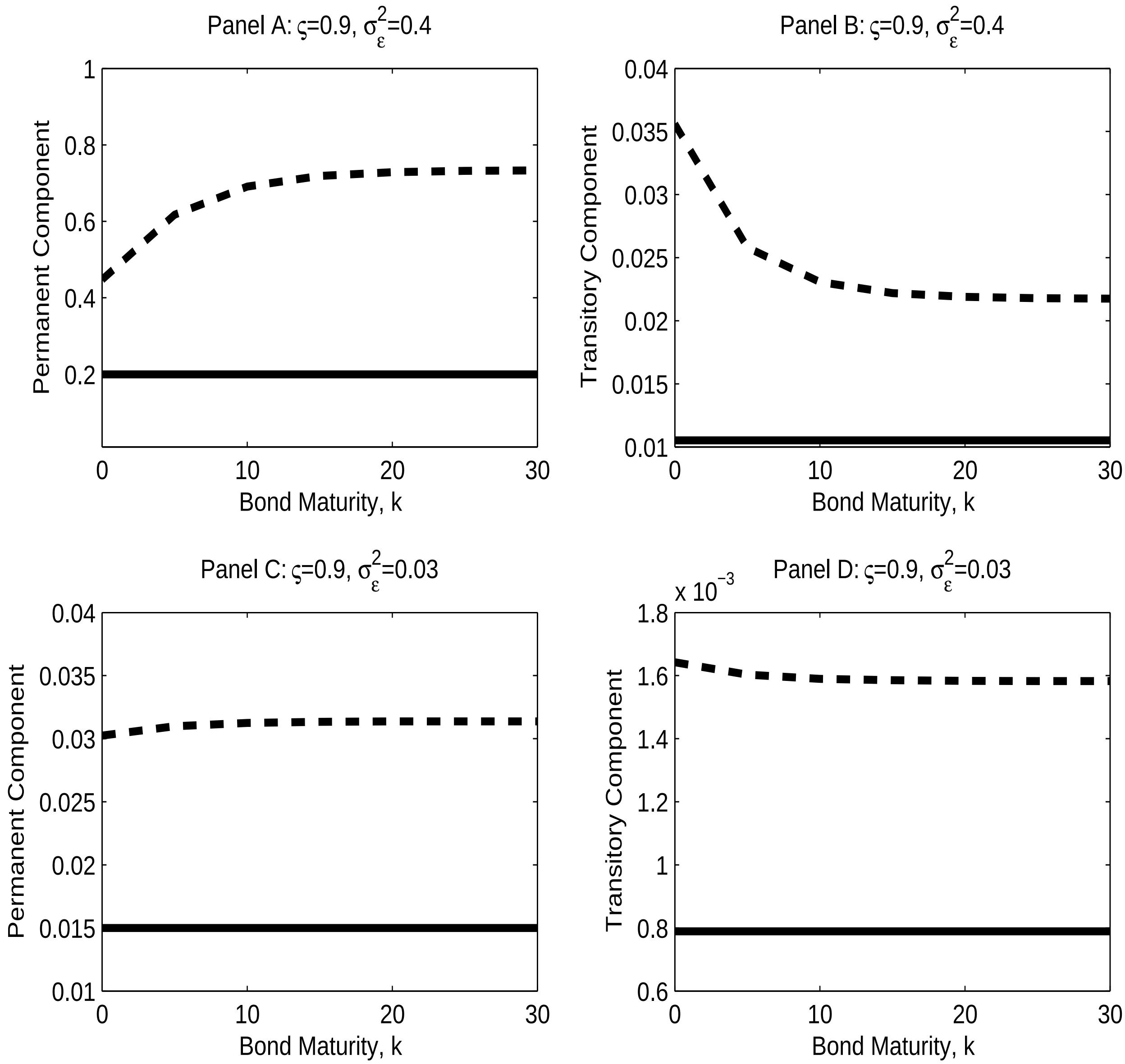

Fig. 1. **Permanent and transitory components from the asset pricing model in Example 2**

Plotted are the unconditional L-measure (solid black curve) and the variance measure (solid dashed curve) corresponding to the permanent and the transitory component of SDF, when the pricing kernel process is $\log(M_{t+1}) = \log(\beta) + \varsigma \log(M_t) + \varepsilon_{t+1}$, with $\varepsilon_{t+1} \sim \mathcal{N}\left(0, \sigma^2_\varepsilon\right)$. The $L\left[\frac{M^P_{t+1}}{M^P_t}\right]$ and $L\left[\frac{M^T_{t+1}}{M^T_t}\right]$, for the permanent and the transitory component, are displayed in equation (23), while $Var\left[\frac{M^P_{t+1}}{M^P_t}\right]$ and $Var\left[\frac{M^T_{t+1}}{M^T_t}\right]$ are displayed in equations (25)–(26). We keep $\beta = 0.998$, $M_0 = 1$, t=1200 months, while the bond maturity, k, is expressed in years.