

A Theory of Asset Prices based on Heterogeneous Investor Information and Limits to Arbitrage*

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Abstract

We propose a theory of asset prices that emphasizes heterogeneous investor beliefs and limits to arbitrage as the main elements determining prices and returns of different securities. In our theory, the noisy aggregation of heterogeneous investor beliefs drives a systematic wedge between the impact of fundamentals on an asset price, and the corresponding impact on cash flow expectations. From this *information aggregation wedge*, we derive testable predictions for prices and returns, both conditional on the realized states, and unconditionally, as a function of an asset's cash flow characteristics and the parameters that summarize the information aggregation frictions. Concretely, assets that are dominated by upside risks trade at a premium, while assets that are dominated by downside risks trade at a discount, on average. These premia and discounts are increasing in the asymmetry in payoff risks and the severity of market frictions. We also use our theory to reconsider the Modigliani-Miller Theorem, the excess volatility puzzle, and the effects of disclosures on market prices.

1 Introduction

We propose a theory of asset prices that emphasizes heterogeneous investor beliefs and limits to arbitrage as the main elements determining prices and returns of different securities. While we are not the first to emphasize the potential role of such frictions for asset pricing, we propose a unified, general framework for studying their joint asset pricing implications. Our main analytical

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innovation consists in formulating a new model of noisy information aggregation through asset prices, which is parsimonious and tractable, yet flexible in the specification of cash flow risks. In our theory, the noisy aggregation of heterogeneous investor beliefs drives a systematic wedge between the impact of fundamentals on an asset price, and the corresponding impact on cash flow expectations. From this *information aggregation wedge*, we derive testable predictions for prices and returns, both conditional on the realized states, and unconditionally, as a function of an asset’s cash flow characteristics and the parameters that summarize the information aggregation frictions. We then use our theory to reconsider three classic results of arbitrage-free asset pricing: the Modigliani-Miller Theorem, the excess volatility puzzle, and the effects of disclosures and transparency for market prices.

We consider an asset market along the lines of Grossman and Stiglitz (1980), Hellwig (1980) and Diamond and Verrecchia (1981), with an investor pool that is divided into informed traders who have observed a noisy signal about the value of an underlying cash flow, and uninformed noise traders. The traders all submit orders to buy shares in the cash flow at the going price. The price serves as a noisy signal of the state, which traders use along with their private signals to form an update about the cash flows. By assuming that traders are risk neutral but face limits on their asset positions, we are able to derive a closed-form characterization for prices and expected dividends conditional on underlying shocks, with no restriction on cash flows other than monotonicity in the underlying fundamental shocks. This allows us to avoid the functional-form restrictions of the canonical models with CARA preferences and normally distributed dividends.

In our model, the asset price is equal to the expectation of cash flows for a “marginal investor” who is just indifferent between investing and not investing in the asset. The information aggregation wedge results from the discrepancy between the marginal traders’ beliefs and the beliefs of an objective, uninformed outsider who uses the observation of the price to update beliefs about dividends, or equivalently an “econometrician” who uses a sample of price-dividend observations to estimate this relationship. Compared to the outsider, the marginal trader appears to treat the information contained in the price as if it was more informative than it truly is. As a result, the marginal trader’s expectations attach a higher weight to the market signal than would be justified by its true information content. Moreover, because of excess confidence in the market signal’s information content, the marginal trader acts as if residual uncertainty was lower than it truly is.

Despite its appearance, the information aggregation wedge is not the result of non-Bayesian updating or irrational trading decisions. Instead, it results from compositional shifts under investor heterogeneity: the identity of the marginal trader changes with the observed price in a way that

amplifies the impact of the price on the marginal trader's expectations. Any trader's expectation of future dividends is based both on his private information and on the information conveyed through the price. The objective expectation of future dividends on the other hand, only reflects the information conveyed by the price. From the perspective of any individual shareholder, the noise in market information is uncorrelated with the noise in private signals, and he therefore treats these two sources of information as independent signals weighted by their respective information content. The marginal trader holds beliefs that make him just indifferent between holding and not holding the asset. A change in fundamentals that is conveyed through the price either shifts the distribution of informed traders' private signals, or restricts the supply of shares that is available to them. In both cases, market-clearing then requires a shift in the marginal traders' identity in the same direction as the market signal. In equilibrium, the market signal is perfectly correlated with the marginal trader's private signal, which gives the asset price the appearance of responding more to the market signal than would be justified by its true information content.

Because the wedge confounds a first-order comparison of fundamental expectations with a second-order comparison of residual uncertainty between the marginal trader's and the outsider's beliefs, sharp comparative statics implications are difficult to obtain for the state-contingent wedge, except in special cases or parametric examples. Intuitively, however, the shift in expectations provides a channel for excess volatility in prices, while the second-order effects affect the unconditional level of prices relative to dividends, by influencing the relative size of the gains and losses that shareholders are exposed to. Our two main theorems make this intuition precise by characterizing unconditional moments of prices, expected dividends and the wedge.

Theorem 1 characterizes the unconditional expectation of prices relative to expected dividends. By forming ex ante expectations, the anticipated first- and second-order effects combine into a mean-preserving spread of the expected marginal trader's weight on different state realizations, relative to the outsider's. This allows us to characterize the unconditional average price and expected dividend in closed form as a function of a parameter summarizing the informational noise of the market, and the profile of the cash flow risk.

What's more, the unconditional wedge has increasing differences between the informational noise parameter, and the asymmetry between upside and downside risks, where the latter is defined as a partial order on payoff risks that compares the marginal gains and losses at fixed distances from the prior mean of the fundamental. This leads to several immediate, general comparative statics results, which form the core asset pricing predictions of our theory. Regardless of the informational noise, the unconditional wedge is zero when payoff risk is symmetric. The wedge is positive (meaning

that the price exceeds expected dividends on average) for risks that are dominated by the upside, and negative for risks that are dominated by the downside. Moreover, in absolute value this wedge becomes more pronounced for more asymmetric payoff risks, or for a higher degree of information aggregation frictions.

Theorem 2 instead focuses on the variability of prices relative to expected and realized dividends. We show that prices are always more variable than expected dividends, and may even be more variable than realized dividends if the information aggregation wedge is sufficiently severe, and in extreme limiting cases, the variability of prices may exceed that of realized dividend by any arbitrary factor. Moreover, the correlation between price and realized dividends may be arbitrarily close to zero.¹

We use our theory to revisit classic results from arbitrage-free asset pricing. In form of the Modigliani-Miller Theorem, the exhaustion of all arbitrage opportunities implies that the total market value of any given cash flow is not influenced by how it is divided into separate securities. Absent distortions inside the firm, a firm's optimal capital structure is indeterminate and disconnected from its market valuation. Capital structure theories then focus mostly on trade-offs inside the firm that affect the generation of cash flows, such as agency costs, information frictions or tax distortions, taking as given that the market value of the resulting cash flow is not affected by its split into different securities.

Consider therefore a seller who is splitting a given cash flow into two pieces which are sold to separate investor pools in two different markets, and suppose that at least one of the pieces is dominated either by upside or by downside risk. The expected revenue of the seller is not affected by the split, if and only if the two markets are characterized by *identical* informational characteristics. When the investor pools differ, the seller can manipulate her expected revenue by selling downside risks in the market with smaller information aggregation frictions, and upside risks in the market with larger information aggregation frictions. The seller maximizes expected revenue by completely separating upside and downside risks, splitting the cash flow into a debt claim for the downside, and an equity claim for the upside, with a default point for debt at the prior mean. All these results follow directly from the increasing difference property. In contrast to capital structure theories that are based on internal frictions, we take cash flows as given, and focus exclusively on investor beliefs

¹To formulate these results for arbitrary payoff risks, we define the variability of a random variable by its expected squared deviation from the variable at the prior mean fundamental, i.e. $\mathbb{E}(P(z) - P(0))^2$ for prices. Similar results can be obtained for other variability measures such as variance or coefficient of variation, but they are less sharp, due to the confounding 2nd-order effects.

and limits to arbitrage, factors that are external to the firm.

Arbitrage-free asset pricing further implies the existence of a common stochastic discount factor that can be used to price all stochastic cash flows. This imposes testable restrictions on the joint evolution of prices and returns for different categories of assets, which serve as a benchmark for understanding observed price fluctuations. For example, the observation of highly volatile asset prices despite seemingly stable underlying cash flows has led to a large literature of risk-based explanations of asset returns to account for the observed high price volatility.

We revisit this “excess volatility puzzle” by considering a simple infinite horizon extension of our static model with a long-lived asset that is traded repeatedly. For a levered equity claim with bounded downside, and unbounded upside risks, the anticipation of positive wedges in future periods leads to higher prices in the current period. Furthermore, the bound on downside risks imposes a floor on the undervaluation of current cash flows. The combination of anticipated future overvaluation with a limit on the current undervaluation implies that the claim may be overvalued in all periods and at all states. Moreover the existence of unbounded upside risks implies that prices may become infinitely volatile. Yet the correlation of prices with the firms’ current fundamentals may be arbitrarily close to zero. These features arise when informational noise is sufficiently severe. They are qualitatively consistent with Shiller’s observation that expected dividends are much less volatile than share prices, and that shares appear to be consistently over-valued, relative to expected dividends, even after periods of large price declines.

Third, with no arbitrage, prices offer an unbiased estimate of risk-adjusted expected future cash flows, conditional on the currently available information. Bayes’ rule then imposes restrictions on the effects of news disclosures on prices. For example, increased transparency, i.e. more accurate information about future cash-flows, always brings prices closer to fundamentals. In general, this also makes market prices more variable.

In our theory, the flip-side of the marginal trader’s excess response to the market price is that he under-reacts to exogenous news about fundamentals. When the information aggregation wedge leads to significant excess price volatility, increased transparency can serve to stabilize prices. At the same time, however, this may also drive prices further away from expected dividends, because prices fail to fully incorporate the information content of disclosures. These effects of transparency are non-monotone: once the transparency is high enough to crowd out the market price as an important information source, the information aggregation wedge disappears, and we converge to arbitrage-free asset pricing.

Finally, we discuss the theoretical robustness of the information aggregation wedge, generalizing

distributional assumptions for fundamentals and signals, allowing noise traders to partially arbitrage the wedge, and discussing the connection with the CARA-normal model. There the wedge exists as well, but the distributional assumptions imposed by the CARA-normal model limit the analysis of the wedge to linear, symmetric risks, which precludes implications for unconditional returns. Any premia or discounts in the average price are based entirely on risk premia.²

Our paper contributes to a large literature on noisy information aggregation in asset markets, including the papers cited above. Our formulation of the asset market draws on Hellwig, Mukherji, and Tsyvinski (2006). The assumption of risk neutrality, which allows us to obtain the general characterization, is both a virtue and a limitation. On the one hand, it induces a form of risk-neutral pricing, which eliminates risk premia as an explanation of prices and therefore highlights the role of limits to arbitrage. On the other hand, this also limits the possibility to integrate our theory with alternative risk-based theories of asset pricing. Remarkably, the information aggregation wedge appears to have received little attention so far, even though it is responsible for many of the rich effects of noisy information aggregation, and helps us unify their interpretation. The only written discussion of this observation that we have found appears in Vives (2008).

Another large literature emphasizes limits to arbitrage and heterogeneous beliefs as potential sources of bubbles, mis-pricing, and other market anomalies (Harrison and Kreps, 1978; Chen, Hong and Stein, 2002; Sheinkman and Xiong, 2003; Hong and Stein, 2007). These models are based on exogenous agents' beliefs (heterogeneous priors) and oftentimes use highly restrictive assumptions on underlying dividends to isolate specific aspects of market behavior. In contrast, we formulate a parsimonious model in the REE tradition in which traders' beliefs update using all available information, including equilibrium prices, according to Bayes rule. The model offers general implications on prices for a rich specification of cash flows, allowing a more systematic discussion about the conditions under which asset prices differ from expected dividends, and how the mispricing depends on the features of the environment or asset characteristics. In this sense, our theoretical contribution pushes us much closer towards offering clear falsifiable predictions for prices and returns on a general class of securities.

In section 2, we describe our model and provide the equilibrium characterization of asset prices. In section 3, we define the information aggregation wedge and discuss at length the resulting prediction for conditional and unconditional asset returns. Section 4 uses the insight offered by

²Because portfolios are linear in posterior beliefs and adjustments take place on the intensive rather than the extensive margin, the reasoning is applied to a shift in the average (not marginal) investor, but the characterization based on a compositional shift is otherwise the same.

these two results to revisit the Modigliani-Miller theorem, the excess volatility puzzle and the effects of public disclosures. Section 5 concludes the analysis with the robustness discussion.

2 Model

2.1 Agents, assets, information structure and financial market

The market is cast as a Bayesian trading game with a unit measure of risk-neutral, informed traders, a stochastic measure of uninformed “noise traders”, and a ‘Walrasian auctioneer’. There is a risky asset whose supply is normalized to a unit measure, and whose dividend is a strictly increasing and twice continuously differentiable function $\pi(\cdot)$ of a stochastic “fundamental” θ .

At the start, nature draws θ according to a normal distribution with mean 0, and unconditional variance σ_θ^2 , $\theta \sim \mathcal{N}(0, \sigma_\theta^2)$. Each informed trader i then receives a noisy private signal x_i which is normally distributed with a mean θ and a variance β^{-1} , and is i.i.d. across traders (conditional on θ), $x_i \sim \mathcal{N}(\theta, \beta^{-1})$. They each decide whether to purchase up to one share of the asset at the prevailing price P , in exchange for cash. Formally, trader i submits a price-contingent demand schedule $d_i(\cdot)$ to maximize her expected wealth $w_i = d_i \cdot (\pi(\theta) - P)$. Traders cannot short-sell the asset or buy additional shares in the market, restricting demand to $[0, 1]$. Individual trading strategies are then a mapping $d : \mathbb{R}^2 \rightarrow [0, 1]$ from signal-price pairs (x_i, P) into asset holdings. Aggregating traders’ decisions leads to the aggregate demand by informed traders, $D : \mathbb{R}^2 \rightarrow [0, 1]$,

$$D(\theta, P) = \int d(x, P) d\Phi(\sqrt{\beta}(x - \theta)), \quad (1)$$

where $\Phi(\cdot)$ denotes a cumulative standard normal distribution, and $\Phi(\sqrt{\beta}(x - \theta))$ represents the cross-sectional distribution of private signals x_i conditional on the realization of θ .³ In addition, there is a stochastic demand for the asset from noise traders, which takes the form $\Phi(u)$, where u is normally distributed with mean zero and variance σ_u^2 , $u \sim \mathcal{N}(0, \sigma_u^2)$, independently of θ . This specification is adapted from Hellwig, Mukherji, and Tsyvinski (2006), and allows us to preserve the tractability of Bayesian updating with normal posterior beliefs.⁴

Once all traders have submitted their orders, the auctioneer selects a price P to clear the market. Formally, the market-clearing price function $P : \mathbb{R}^2 \rightarrow \mathbb{R}$ selects P from the correspondence $\hat{P}(\theta, u) = \{P \in \mathbb{R} : D(\theta, P) + \Phi(u) = 1\}$, for all $(\theta, u) \in \mathbb{R}^2$.⁵ After all trades are completed, the

³We assume that the Law of Large Numbers applies to the continuum of traders, so that conditional on θ the cross-sectional distribution of signal realizations ex post is the same as the ex ante distribution of traders’ signals.

⁴We generalize this demand specification in Section 4.2 allowing for price-elastic demands by noise traders.

⁵We can without loss of generality restrict the range of $P(\cdot)$ to coincide with the range of $\pi(\cdot)$.

dividends $\pi(\theta)$ are realized and disbursed to the owners of the asset.

Let $H(\cdot|x, P) : \mathbb{R} \rightarrow [0, 1]$ denote the traders' posterior cdf of θ , conditional on observing a private signal x , and conditional on the market price P . A *Perfect Bayesian Equilibrium* consists of demand functions $d(x, P)$ for informed traders, a price function $P(\theta, u)$, and posterior beliefs $H(\cdot|x, P)$ such that (i) $d(x, P)$ is optimal given $H(\cdot|x, P)$; (ii) the asset market clears for all (θ, u) ; and (iii) $H(\cdot|x, P)$ satisfies Bayes' rule whenever applicable, i.e., for all p such that $\{(\theta, u) : P(\theta, u) = p\}$ is non-empty.

2.2 Equilibrium Characterization

We begin by characterizing informed traders' demand. With risk-neutrality, the trader's expected value of holding the asset is $\int \pi(\theta)dH(\theta|x, P)$. Since private signals are log-concave and $\pi(\cdot)$ is increasing in θ , posterior beliefs $H(\cdot|x, P)$ are first-order stochastically increasing in x , and $\int \pi(\theta)dH(\theta|x, P)$ is strictly increasing in x , for any P that is observed in equilibrium. The traders' decisions are therefore characterized by a signal threshold function $\hat{x} : \mathbb{R} \rightarrow \mathbb{R} \cup \{\pm\infty\}$, such that $d(x_i, P) = \mathbb{I}_{x_i \geq \hat{x}(P)}$, that is, the trader places an order to buy a share at price P , if and only if $x_i \geq \hat{x}(P)$. The trader who observes the signal equal to the threshold, $x = \hat{x}(P)$, and who is therefore indifferent, is called the *marginal trader*. The signal threshold is uniquely defined by

$$\begin{aligned} \hat{x}(P) &= +\infty \quad \text{if } \lim_{x \rightarrow +\infty} \int \pi(\theta)dH(\theta|x, P) \leq P, \\ \hat{x}(P) &= -\infty \quad \text{if } \lim_{x \rightarrow -\infty} \int \pi(\theta)dH(\theta|x, P) \geq P, \\ P &= \int \pi(\theta)dH(\theta|\hat{x}(P), P) \quad \text{otherwise.} \end{aligned} \tag{2}$$

Expression (2) illustrates three cases: (i) if the most optimistic trader's expected dividend is lower than the price, no trader buys so the signal threshold becomes $+\infty$; (ii) if the most pessimistic trader's expected dividend exceeds the price, all traders buy and the threshold for buying is $-\infty$; (iii) only some traders buy, and the threshold $\hat{x}(P)$ takes an interior value at which the marginal trader's posterior expectation of the dividend must equal the price. Aggregating the individual trading decisions, the informed demand is $D(\theta, P) = \int_{\hat{x}(P)}^{\infty} 1 \cdot d\Phi(\sqrt{\beta}(x - \theta)) = 1 - \Phi(\sqrt{\beta}(\hat{x}(P) - \theta))$, which equals 0 if $\hat{x}(P) = +\infty$, and 1 if $\hat{x}(P) = -\infty$.

Next, we analyze the market-clearing condition. Since $\Phi(u) \in (0, 1)$, in equilibrium, $\hat{x}(\cdot)$ must be finite for all P on the equilibrium path, and satisfy the third condition in (2). From the market-clearing condition, we then have $\Phi(\sqrt{\beta}(\hat{x}(P) - \theta)) = \Phi(u)$, which allows us to characterize the

correspondence of market-clearing prices:

$$\hat{P}(\theta, u) = \left\{ P \in \mathbb{R} : \hat{x}(P) = \theta + \frac{1}{\sqrt{\beta}}u \right\}. \quad (3)$$

From now on, we focus on equilibria in which the price is conditioned on (θ, u) through the observable “state” variable $z \equiv \theta + 1/\sqrt{\beta} \cdot u$. The next lemma characterizes the resulting equilibrium beliefs. All proofs are provided in the appendix.

Lemma 1 (Information Aggregation) *(i) In any equilibrium with conditioning on z , the equilibrium price function $P(z)$ is invertible. (ii) Equilibrium beliefs for price realizations observed along the equilibrium path are given by*

$$H(\theta|x, P) = \Phi \left(\frac{\theta - \frac{\beta x + \beta \sigma_u^{-2} \cdot \hat{x}(P)}{\sigma_\theta^{-2} + \beta + \beta \sigma_u^{-2}}}{\sqrt{\sigma_\theta^{-2} + \beta + \beta \sigma_u^{-2}}} \right). \quad (4)$$

Part (ii) of the Lemma exploits the invertibility to arrive at a complete characterization of posterior beliefs $H(\cdot|x, P)$. With invertibility, we can summarize information conveyed by the price through z . Conditional on θ , z is normally distributed with mean θ and variance σ_u^2/β . Thus, the price is isomorphic to a normally distributed signal of θ , with a precision that is increasing in the precision of private signals, and decreasing in the variance of demand shocks.

Using Lemma 1 we rewrite (2), the indifference condition that defines the signal threshold $\hat{x}(P)$:

$$P = \int \pi(\theta) d\Phi \left(\frac{\theta - \frac{\beta + \beta \sigma_u^{-2}}{\sigma_\theta^{-2} + \beta + \beta \sigma_u^{-2}} \hat{x}(P)}{\sqrt{\sigma_\theta^{-2} + \beta + \beta \sigma_u^{-2}}} \right). \quad (5)$$

This condition equates P to the marginal trader’s expectation of dividends. The latter also depends on P through its effect on posterior beliefs. Using the market-clearing condition $\hat{x}(P) = z$, Proposition 1 uniquely characterizes the market equilibrium.⁶

Proposition 1 (Asset market equilibrium) *For any increasing dividend function $\pi(\cdot)$, an asset market equilibrium exists, is unique, and characterized by the price function $P_\pi(z)$ and the traders’ threshold function $\hat{x}(p) = z = P_\pi^{-1}(p)$, where*

$$P_\pi(z) = \mathbb{E}(\pi(\theta)|x = z, z) = \int \pi(\theta) d\Phi \left(\frac{\theta - \frac{\beta + \beta \sigma_u^{-2}}{\sigma_\theta^{-2} + \beta + \beta \sigma_u^{-2}} z}{\sqrt{\sigma_\theta^{-2} + \beta + \beta \sigma_u^{-2}}} \right). \quad (6)$$

The price function $P_\pi(z)$ is uniquely defined and strictly monotone, and therefore defines the unique market equilibrium.⁷

⁶Notice that this only implies the uniqueness of the equilibrium that conditions on the summary statistic z , not overall uniqueness of the equilibrium characterized in proposition 1.

⁷We’ll index n equilibrium function or variable by π to make explicit that it is derived from a specific dividend function $\pi(\cdot)$, i.e. $P_\pi(\cdot)$ is the equilibrium price function that is derived from dividend function $\pi(\cdot)$ by equation 6.

3 The Information Aggregation Wedge

3.1 Conditional Information aggregation wedge

We now discuss how noisy information affects equilibrium prices and expected dividend values. To be precise, we form expectations of dividends from the perspective of an outside observer (or “econometrician”) who does not have access to any private signal about θ , but knows the parameters of the game and observes the realization of the price P , or equivalently the state z . This outsider holds a conditional belief that $\theta|z \sim \mathcal{N}(\beta\sigma_u^{-2}/(\sigma_\theta^{-2} + \beta\sigma_u^{-2}) \cdot z, \sigma_\theta^2 + \sigma_u^2/\beta)$, and therefore has an expectation of dividends conditional on public information z , denoted $V_\pi(z)$, which is equal to

$$V_\pi(z) = \mathbb{E}(\pi(\theta)|z) = \int \pi(\theta) d\Phi \left(\sqrt{\sigma_\theta^{-2} + \beta\sigma_u^{-2}} \left(\theta - \frac{\beta\sigma_u^{-2}}{\sigma_\theta^{-2} + \beta\sigma_u^{-2}} z \right) \right). \quad (7)$$

The main observation from comparing Proposition 1 with equation (7) is that at the interim stage –when the share price is observed but before dividends are realized– the equilibrium price differs from the expected dividend, conditional on the public information. This difference is due to the impact of private information on equilibrium prices. We label this difference the *information aggregation wedge*, $W_\pi(z) \equiv P_\pi(z) - V_\pi(z)$.

The price $P_\pi(z)$, and the expected dividend conditional on public information, $V_\pi(z)$, differ in how expectations of θ are formed. The price equals the dividend expectation of the marginal trader who is indifferent between keeping or selling her share. This trader conditions on the market signal z , as well as a private signal, whose realization must equal the threshold $\hat{x}(P)$ in order to be consistent with the trader’s indifference condition. The trader treats these two sources of information as mutually independent signals of θ . At the same time, the market-clearing condition implies that $\hat{x}(P)$ must equal z in order to equate demand and supply of shares. The marginal trader’s expectation $\mathbb{E}(\pi(\theta)|x = z, z)$ thus behaves as if she received one signal z of precision $\beta + \beta\sigma_u^{-2}$ instead of $\beta\sigma_u^{-2}$. In contrast, the expected dividends $\mathbb{E}(\pi(\theta)|z)$ conditional on P (or equivalently z) weighs z according to its true precision $\beta\sigma_u^{-2}$.

Alternatively, we can view the difference in the responsiveness of price and the expected dividend conditional on the price, as the result of a compositional change in the identity of the traders holding the shares. An increase in z raises the price and expected dividend through a direct effect on all traders’ expectations. This is reflected in the weight $\beta\sigma_u^{-2}$ attributed to z in both $P_\pi(z)$ and $V_\pi(z)$. In addition, an increase in z changes the identity of the marginal trader: since the random demand for shares is larger (on average) for a higher z , market clearing requires more selling from the original shareholders. This implies that the new marginal trader must have higher expectations

about the dividend, which holds in equilibrium since her private signal now corresponds to the higher realization of z .⁸ The resulting extra shift in prices is captured by the additional weight β attributed to z in the price function.

Belief heterogeneity and limits to arbitrage are both necessary ingredients to obtain the wedge. If instead all informed traders have access to a common signal z of fundamentals, they all hold identical expectations and must be indifferent between buying and not buying to clear the market. But this requires that the price equals the common expectation of the dividend, i.e. $P_\pi(z) = V_\pi(z)$. The same result applies with free entry of uninformed arbitrageurs (Kyle, 1985).

The remainder of this subsection describes properties of the wedge, conditional on z , which will form the basis for our main results on expected returns and price volatility. To this end, we define

$$\gamma_P = \frac{\beta + \beta\sigma_u^{-2}}{\sigma_\theta^{-2} + \beta + \beta\sigma_u^{-2}} \text{ and } \gamma_V = \frac{\beta\sigma_u^{-2}}{\sigma_\theta^{-2} + \beta\sigma_u^{-2}}$$

as the response coefficients of $P_\pi(\cdot)$ and $V_\pi(\cdot)$ to innovations in z . The equilibrium price and expected dividends are then rewritten as:

$$\begin{aligned} V_\pi(z) &= \int \pi(\gamma_V z + \sigma_\theta \sqrt{1 - \gamma_V} u) \phi(u) du \\ P_\pi(z) &= \int \pi(\gamma_P z + \sigma_\theta \sqrt{1 - \gamma_P} u) \phi(u) du \end{aligned}$$

This formulation summarizes the difference between the price and the expected dividend by the response parameters γ_P and γ_V which measure the marginal trader's and outsider's update to z . These parameter enter $P_\pi(\cdot)$ and $V_\pi(\cdot)$ in two ways: the marginal trader's expectation responds more strongly to z , and his residual uncertainty θ is lower - $\sigma_\theta^2(1 - \gamma_P)$ instead of $\sigma_\theta^2(1 - \gamma_V)$. From an ex ante perspective, $z \sim \mathcal{N}(0, \sigma_\theta^2/\gamma_V)$. Using a third-order Taylor expansion, we approximate the wedge by

$$W_\pi(z) \approx \pi(\gamma_P z) - \pi(\gamma_V z) + \frac{\sigma_\theta^2}{2} [\pi''(\gamma_P z)(1 - \gamma_P) - \pi''(\gamma_V z)(1 - \gamma_V)]$$

Here, the term $\pi(\gamma_P z) - \pi(\gamma_V z)$ captures the shift in expectations, while the second term in squared brackets captures the role of residual uncertainty. The latter plays a role only if $\pi(\cdot)$ is non-linear, and in that case matters through second- and higher derivatives. The shift in expectations from $\gamma_V z$ to $\gamma_P z$ amounts to a mean-preserving spread from an ex ante perspective, and is therefore a

⁸If z increases because of θ , the distribution of private signals shifts up, decreasing supply for a given P . If instead z increases because of u , the distribution of signals and hence supply remains unchanged, but the demand for shares has gone up. Of course, in equilibrium traders cannot disentangle these two possibilities from observing the price.

source of increased variability in the price, relative to expected fundamentals: $\pi(\gamma_P z) - \pi(\gamma_V z)$ crosses 0 at a single point, where $z = 0$, is negative when $z < 0$, and positive when $z > 0$.

When $\pi(\cdot)$ is linear, this is the only effect entering the analysis. For the linear dividend $\pi(\theta) = \theta$, panel a) of figure 1 plots the price (solid line), the expected dividend (dashed line) and conditional wedge (dashed-dot line) as a function of the state variable z . The price is more sensitive to innovations in z than the expected dividend, resulting in a wedge $W_\pi(z) = (\gamma_P - \gamma_V)z$ that is negative for $z < 0$, zero for $z = 0$, and positive for $z > 0$. This excess sensitivity of the wedge results because only the higher sensitivity of expectations matters for the characterization, while the residual uncertainty plays no role.

Residual uncertainty, on the other hand, shifts the level of the wedge up or down, depending on a comparison between the residual uncertainty levels $1 - \gamma_P$ relative to $1 - \gamma_V$, and the second derivatives $\pi''(\gamma_P z)$ and $\pi''(\gamma_V z)$. At the prior mean $z = 0$, the second derivatives are comparable, so the reduction of uncertainty implies a negative wedge if $\pi''(0) > 0$, and a positive wedge, if $\pi''(0) < 0$. Away from $z = 0$, the third- and higher derivatives may reduce or even overturn this effect, and therefore make it impossible to offer precise results on the shape of $W_\pi(\cdot)$ without additional restrictions. The following cubic example shows that $W_\pi(\cdot)$ need not be monotone, and it may also cross 0 at multiple points.

Example 1: Suppose that $\pi(\theta) = \theta + a\theta^3$, with $a > 0$ to ensure monotonicity of π . For a cubic function, the above approximation holds exactly, so that

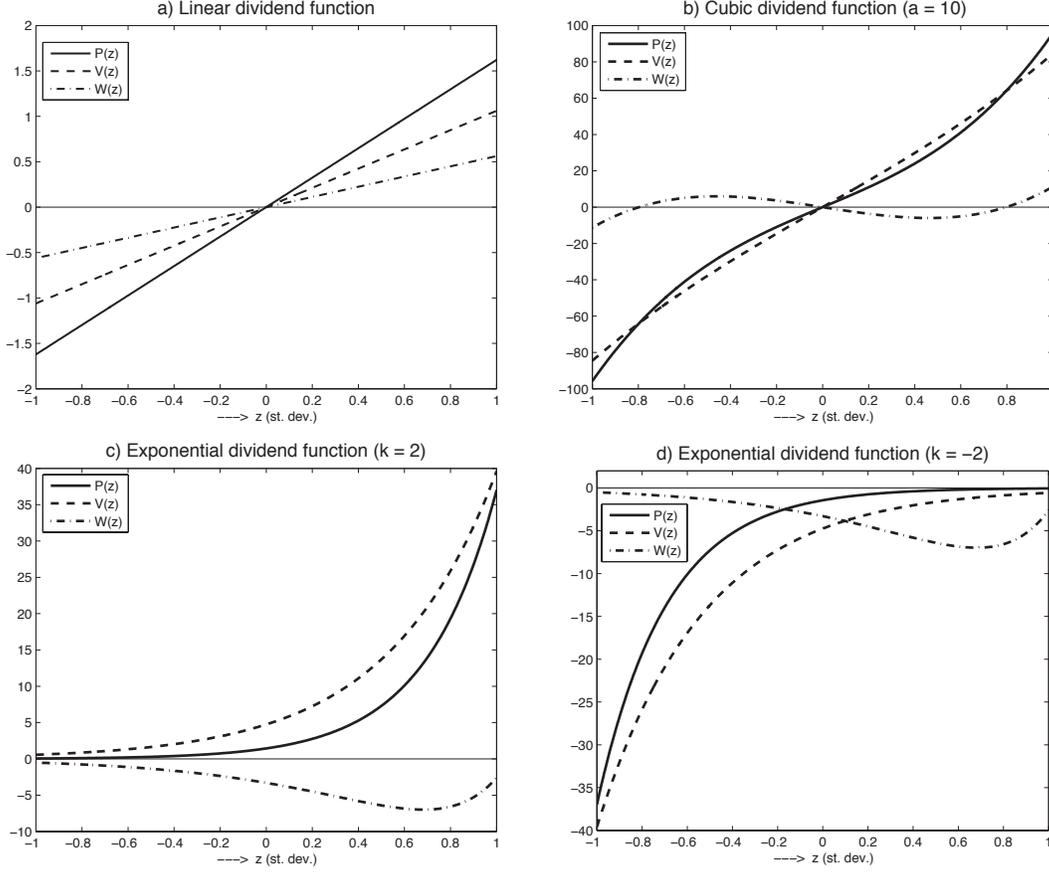
$$W_\pi(z) = (\gamma_P - \gamma_V)z + a(\gamma_P^3 - \gamma_V^3)z^3 + 3az\sigma_\theta^2 [\gamma_P(1 - \gamma_P) - \gamma_V(1 - \gamma_V)]$$

where the first two terms correspond to the shift in means, and the last to the shift in residual uncertainty. If $\gamma_P + \gamma_V > 1$ and a sufficiently large, $W'_\pi(0) < 0$. Moreover, we also have $W_\pi(0) = 0$, and it therefore follows immediately that $W_\pi(\cdot)$ is non-monotone and crosses 0 in three different locations.

Our second example, plotted in panel b) of figure 1, offers a case in which all the insights from the shifts in means and residual uncertainty hold more generally. We will use this example to illustrate the main theoretical results on expected prices and price volatility. The challenge for the remainder will be to establish that these results hold more generally.

Example 2: Suppose that $\pi(\theta) = \frac{1}{k}e^{k\theta}$, with $k \neq 0$. Then, expected dividends, prices and the

Figure 1: Conditional Price, Expected Dividend and Wedge



wedge are characterized by:

$$V_\pi(z) = \frac{1}{k} e^{k\gamma_V z + \frac{k^2}{2} \sigma_\theta^2 (1-\gamma_V)}, \quad P_\pi(z) = \frac{1}{k} e^{k\gamma_P z + \frac{k^2}{2} \sigma_\theta^2 (1-\gamma_P)}$$

$$W_\pi(z) = P_\pi(z) \left(1 - e^{-k(\gamma_P - \gamma_V)z + \frac{k^2}{2} (\gamma_P - \gamma_V) \sigma_\theta^2} \right).$$

In this case, the price and expected dividend are both exponential functions in z , with a stronger reaction of prices to z . The residual uncertainty affects both $V_\pi(z)$ and $P_\pi(z)$ multiplicatively, but the factor is larger for $V_\pi(z)$, reflecting the fact that residual uncertainty is greater for the outsider.

If $k > 0$, we then have a dividend function that is increasing, convex, and bounded below by zero (figure 1, panel c). The wedge is negative at $z = 0$, but increasing and convex in z , for $z > 0$ and crossing 0 at $z = \frac{k}{2} \sigma_\theta^2 > 0$. For $z < 0$, $W_\pi(z)$ is negative, and non-monotone, decreasing at first, reaching its lowest value at some intermediate point, and increasing from there on. The reverse image obtains when $k < 0$, in which case π is increasing, concave, and bounded above by zero (figure

1, panel d). The wedge then is positive at $z = 0$, and for negative values, $W_\pi(\cdot)$ is increasing and concave (crossing 0 at $z = \frac{k}{2}\sigma_\theta^{-2} < 0$). For $z > 0$, $W_\pi(\cdot)$ is at first increasing and then decreasing and converging to 0 after reaching an interior maximum. This example thus confirms the intuitions from the shift in means which makes $P_\pi(z)$ more responsive to a shift in z , and the shift in residual uncertainty that is captured by the multiplicative factors. The curvature parameter k governs the shape of the wedge function, and whether the residual uncertainty increases or decreases the wedge.

We use this example to illustrate our two main results. First, we show that the expectation of the expected wedge is positive if and only if $k > 0$, and negative if $k < 0$. That is, the security trades at a premium in the case with convex dividends and upside risks, and at a discount in the case with concave dividends and downside risks. Taking expectations, we have

$$\begin{aligned} \mathbb{E}(V_\pi(z)) &= 1/k \cdot e^{\frac{k^2}{2}\sigma_\theta^2}, & \mathbb{E}(P_\pi(z)) &= 1/k \cdot e^{\frac{k^2}{2}\sigma_\theta^2} [1 + (\frac{\gamma_P}{\gamma_V} - 1)\gamma_P] \\ \text{and } \mathbb{E}(W_\pi(z)) &= 1/k \cdot e^{\frac{k^2}{2}\sigma_\theta^2} \left\{ e^{\frac{k^2}{2}\sigma_\theta^2(\gamma_P/\gamma_V - 1)\gamma_P} - 1 \right\}, \end{aligned}$$

which is positive, whenever $k > 0$, and negative, whenever $k < 0$ (and can be checked to approach 0 continuously as $k \rightarrow 0$).

Second, we show that the model exhibits excess price volatility. Focusing on log variances for analytical convenience, we have: $Var(\log \pi(\theta)) = k^2\sigma_\theta^2$, $Var(\log V_\pi(z)) = \gamma_V k^2\sigma_\theta^2$, and $Var(\log P_\pi(z)) = \gamma_P^2/\gamma_V \cdot k^2\sigma_\theta^2$. Therefore, we observe that $Var(\log P_\pi(z)) > Var(\log V_\pi(z))$, for any parameter set. Moreover, if the information aggregation wedge becomes sufficiently important, then we may have $\gamma_P^2/\gamma_V > 1$, and therefore $Var(\log P_\pi(z)) > Var(\log \pi(\theta))$. This results in particular in two limiting scenarios: (i) if for given $\gamma_V < 1$, γ_P approaches 1, i.e. the informed traders have very precise signals for given level of information in the price, or if $\gamma_V \rightarrow 0$, while γ_P is bounded away from 0. In this case, the market becomes very noisy, for a given level of private information. On the other hand, the $Var(\log V_\pi(z))$ is always less than $Var(\log \pi(\theta))$, which is a standard result for Bayesian updating with noisy signals (Blackwell's theorem).

Our main two theorems generalize the observations from these examples. Theorem 1 below establishes that the sign and magnitude of the average wedge on the comparison of upside vs. downside risks. Theorem 2 generalizes the result that prices are more variable than expected dividends, and in some cases even more variable than realized dividends.

3.2 Unconditional information aggregation wedge

We obtain much sharper results for the unconditional expected information aggregation wedge. Let $W_\pi = \mathbb{E}(W_\pi(z))$ denote the expected information aggregation wedge associated with a payoff

function $\pi(\cdot)$. The next lemma provides a characterization of W_π which forms the basis for the subsequent comparative statics results.

Lemma 2 (Unconditional Wedge) *Define σ_P as $\sigma_P^2 = \sigma_\theta^2(1 + (\gamma_P/\gamma_V - 1)\gamma_P)$. The unconditional information aggregation wedge W_π is characterized by*

$$W_\pi(\sigma_P) = \int_0^\infty (\pi'(\theta) - \pi'(-\theta)) \left(\Phi\left(\frac{\theta}{\sigma_\theta}\right) - \Phi\left(\frac{\theta}{\sigma_P}\right) \right) d\theta.$$

This characterization allows us to clearly separate the effects of the payoff function or its derivative, from the informational parameters, which are summarized by $\sigma_P > \sigma_\theta$, a parameter that summarizes the importance of informational frictions in the market. By taking ex ante expectations over z , the shifts in mean and residual uncertainty combine into a mean-preserving spread of the weights associated with different realizations of θ . From an ex ante perspective, the marginal trader places more weight on the tails of the fundamental distribution. The parameter σ_P measures the importance of this mean-preserving spread, and fully summarizes the impact of the informational environment on the market price. This result can intuitively be understood as follows: the marginal trader's posterior of θ , conditional on z is normal with mean $\gamma_P z$ and variance $(1 - \gamma_P)\sigma_\theta^2$. The prior over z on the other hand is normal with mean 0 and variance σ_θ^2/γ_V . Therefore compounding the two distributions, the marginal trader's prior over θ is characterized as a normal distribution with mean 0 and variance equal to $(1 - \gamma_P)\sigma_\theta^2 + \gamma_P^2\sigma_\theta^2/\gamma_V = \sigma_P^2$. The outsider on the other hand holds a posterior that θ , conditional on z is normal with mean $\gamma_V z$ and variance $(1 - \gamma_V)\sigma_\theta^2$, his prior over θ is $(1 - \gamma_V)\sigma_\theta^2 + \gamma_V\sigma_\theta^2 = \sigma_\theta^2$.

We use this to sign the wedge as a function of the shape of the dividend function, and to offer comparative statics with respect to π and the informational parameters γ_P and γ_V . Our next definition provides a partial order on payoff functions that we will use for the comparative statics.

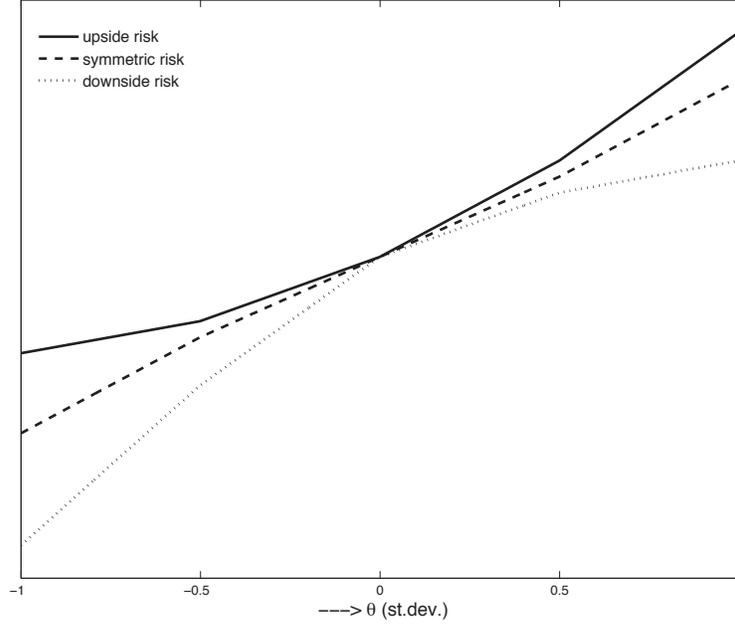
Definition 1 (i) *A dividend function π has symmetric risks if $\pi'(\theta) = \pi'(-\theta)$ for all $\theta > 0$.*

(ii) *A payoff function π is dominated by upside risks, if $\pi'(\theta) \geq \pi'(-\theta)$ for all $\theta > 0$. A payoff function π is dominated by downside risks, if $\pi'(\theta) \leq \pi'(-\theta)$ for all $\theta > 0$.*

(iii) *A dividend function π_1 has more upside and less downside risk than π_2 if $\pi_1'(\theta) - \pi_1'(-\theta) \geq \pi_2'(\theta) - \pi_2'(-\theta)$ for all $\theta > 0$.*

This definition offers a criterion for comparing the asymmetries between different payoff functions, and classifies them by comparing marginal gains and losses at fixed distances from the prior mean to determine whether the payoff exposes its owner to bigger payoff fluctuations on the upside or the downside. Any linear dividend function has symmetric risks, any convex function is

Figure 2: Dividend risk types



dominated by upside risks, and any concave dividend function is dominated by downside risks. The classification however also extends to non-linear functions with symmetric gains and losses, as well as non-convex functions with upside risk or non-concave functions with downside risk. The following Theorem summarizes the comparative statics implications that follow directly from this partial order, and the characterization in lemma 2.

Theorem 1 (Average prices and returns) (i) *Sign:* If π has symmetric risk, then $W_\pi = 0$. If π is dominated by upside risk, then $W_\pi \geq 0$. If π is dominated by downside risk, then $W_\pi \leq 0$.

(ii) *Comparative Statics w.r.t. π :* For given σ_P^2 , if π_1 has more downside and less upside risk than π_2 , then $W_{\pi_2} \geq W_{\pi_1}$.

(iii) *Comparative Statics w.r.t. σ_P^2 :* If π is dominated by upside or downside risk, then $\|W_\pi\|$ is increasing in σ_P . Moreover, $\lim_{\sigma_P \rightarrow \sigma_\theta} \|W_\pi\| = 0$.

(iv) *Increasing differences:* If π_1 has more downside and less upside risk than π_2 , then $W_{\pi_2}(\sigma_P) - W_{\pi_1}(\sigma_P)$ is increasing in σ_P .

This theorem summarizes how the shape of the dividend function and the informational parameters combine to determine the sign and magnitude of the unconditional information aggregation

wedge. It shows that unconditional price premia or discounts arise as a combination of two elements: upside or downside risks in the dividend profile π , and an impact of private information on market prices ($\gamma_P > \gamma_V$). The latter requires that updating from prices is noisy ($\gamma_V < 1$). This Theorem forms the first part of our core theoretical contribution, and shows that noisy information aggregation may influence conditional and unconditional returns of assets through their payoff profile and the informational characteristics of the market.

The result is easily understood from our interpretation of the wedge as the expected value of a symmetric, mean-preserving spread of the true underlying fundamental distribution.

Part (i) shows that the sign of the wedge is determined by whether π is dominated by upside risk, downside risk, or symmetric. When the dividend function has symmetric risk, the gains from this spread on the upside exactly cancel the expected losses on the downside, and the total effect is 0 (part (i)), when the dividend is dominated by upside risks, the expected upside gains dominate, and hence the value of the mean-preserving spread is positive, while when the expected dividend is dominated by downside risks, the expected losses on the downside dominate, and hence the expected value of the spread is negative.

Parts (ii), (iii), and (iv) complement the first result on the possibility of price premia or discounts with specific predictions on how its magnitude depends on cash flow and informational characteristics.

Part (ii) shows that an asset with more upside or less downside risk will on average have a higher price premium or a lower price discount, all else equal. Thus, returns will on average be lower (and prices higher) for securities that represent more upside risks. Simply put, the mean-preserving spread becomes more valuable when the payoff function shifts towards more upside risk.

Part (iii) shows the role of informational parameters. For a given payoff, the wedge increases in absolute value as the information aggregation friction has bigger effects (higher σ_P). For a given set of upside or downside risks, a bigger mean-preserving spread generates bigger gains or losses.

Moreover, a wedge obtains only if $\gamma_P > \gamma_V$, i.e. if the heterogeneous beliefs have an impact on price, and the wedge is increasing in γ_P and decreasing in γ_V , that is precision of market information and precision of private information move the wedge in opposite directions. The absolute wedge becomes largest, and approaches infinity, when $\gamma_V \rightarrow 0$. This obtains, if for a given value of β , the market noise becomes infinitely large. In this limiting case, the marginal trader remains responsive to z , even though the z is infinitely noisy.

Part (iv) shows that the average wedge has increasing differences between the dominance of upside risk and the level of market noise. This implies that the effects of market noise and asym-

metry in dividend risk are mutually reinforcing on the magnitude of the wedge. In fact all the other results can be seen as direct consequences of a zero wedge for symmetric risks, and the increasing difference property.

Importantly, our results on differences between expected prices and dividends are not a consequence of irrational trading strategies, behavioral biases of investors, or agency conflicts. Nor are such differences accounted for by risk premia (since traders are risk neutral). Our model thus offers a theory in which average prices can differ systematically from expected dividends can generally differ as a result of the interplay between the dividend structure and the partial aggregation of information into prices, in a context where traders hold heterogeneous beliefs in equilibrium and arbitrage is limited. To our knowledge, this channel is new to the literature.

3.3 Excess Price Variability

Our second main result concerns the variability of prices, relative to expected dividends and realized dividends. As can readily be seen from the above characterizations, if $W'_\pi(\cdot) > 0$, it can be seen immediately that the unconditional variance of prices (prior to realization of z) exceeds the variability of expected dividends. Consider furthermore the limiting case where $\gamma_P \rightarrow 1$, in which $P_\pi(z) \rightarrow \pi(z)$. Since the variance of z exceeds that of θ , it follows immediately that in this limit, where the informed traders' signals become arbitrarily precise, the variability of prices can exceed the variability of dividends. The same observations also arose immediately from the linear and the log-normal models.

Our second theorem generalizes these observations. To do so, we will need to impose some restrictions to handle the non-linearities and higher-order effects that are confounding the comparative statics of W_π w.r.t. z . Concretely, we will focus on risks that are symmetric or dominated by the upside or downside, and we will focus on $\mathbb{E}\left((P_\pi(z) - P_\pi(0))^2\right)$, $\mathbb{E}\left((V_\pi(z) - V_\pi(0))^2\right)$ and $\mathbb{E}\left((\pi(\theta) - \pi(0))^2\right)$ as our criterion for the variability of prices, expected dividends, and realized dividends, respectively, rather than the unconditional variances. The next theorem states our main result concerning excess variability:

Theorem 2 (Excess variability of prices) *For any payoff function $\pi(\cdot)$ that is symmetric, dominated by upside, or dominated by downside risk:*

(i) *The variability of expected dividends is always less than the variability or realized dividends and the variability of prices:*

$$\mathbb{E}\left((V_\pi(z) - V_\pi(0))^2\right) < \mathbb{E}\left((\pi(\theta) - \pi(0))^2\right) \quad \text{and} \quad \mathbb{E}\left((V_\pi(z) - V_\pi(0))^2\right) < \mathbb{E}\left((P_\pi(z) - P_\pi(0))^2\right)$$

(ii) *The excess variability of prices relative to expected dividends is increasing in γ_P and decreasing in γ_V .*

(iii) *For any γ_V , if γ_P is sufficiently high, then the variability of prices exceeds the variability of realized dividends. The same occurs if, for given γ_P , γ_V is sufficiently low.*

(iv) *If $\pi(\cdot)$ is unbounded on one side, then $\lim_{\gamma_V \rightarrow 0} \mathbb{E} \left((P_\pi(z) - P_\pi(0))^2 \right) = \infty$*

This theorem summarizes our findings regarding price variability, and shows that in the limit where the market becomes very noisy, price variability may become arbitrarily large, and even exceed the variability of realized dividends. This occurs because the response of prices to z is bounded away from 0, even if prices become infinitely noisy. At the other extreme, we also obtain excess price variability (beyond the variability of realized dividends), if the private signals become arbitrarily precise.

The statement of the result relies on two restrictions which we used for analytical tractability. First, the focus on a variability measure which combines a variance with a bias between the average price and the price at the average fundamental. This formulation is the one giving us easily interpretable results. In the proof, we break down the comparison between $\mathbb{E} \left((P_\pi(z) - P_\pi(0))^2 \right)$ and $\mathbb{E} \left((V_\pi(z) - V_\pi(0))^2 \right)$ into two parts: first, we compare $\mathbb{E} \left((P_\pi(z) - P_\pi(0))^2 \right)$ where the expectations are taken with respect to the true distribution of z , $z \sim \mathcal{N}(0, \sigma_\theta^2/\gamma_V)$ with the alternative expectations taken under the distribution $z \sim \mathcal{N}(0, \sigma_\theta^2/\gamma_P)$, under which the market's expectations are consistent with Bayes' rule. This offers us a measure of the variability coming from the over-reaction to z . Second, we consider the comparison between $\mathbb{E} \left((P_\pi(z) - P_\pi(0))^2 \right)$ under the distribution $z \sim \mathcal{N}(0, \sigma_\theta^2/\gamma_P)$, and $\mathbb{E} \left((V_\pi(z) - V_\pi(0))^2 \right)$ under the distribution $z \sim \mathcal{N}(0, \sigma_\theta^2/\gamma_V)$. This second comparison measures the effects of having less residual uncertainty about θ . In effect, we are doing the comparison by inter-posing between the marginal trader and our uninformed outsider, a third person who observes signal z that has the same precision as the information observed by the marginal trader. This third person is fully Bayesian, like the outsider, but has information of the same precision as the marginal trader.

We then show that these two terms have clear comparative statics implications. The first, over-reaction effect always increases the variability of prices, and enough to generate price volatility in excess of the variability of dividends. The second comparison can be broken down into a comparison of variances of $P_\pi(z)$ and $V_\pi(z)$ under their respective distributions for z , and a comparison of biases. The comparison of variances follows directly using Blackwell's comparisons for information content of signals. To compare the biases, we use the fact that we can easily order them for

distributions that are symmetric, or dominated by upside or downside risk. This is the only place where the restriction to dividends that are dominated by upside or downside risk plays a role. Similar results can be shown for variances or variability more generally, if one is willing to impose other restrictions on payoffs (to guarantee that $W'_\pi(\cdot) > 0$, for example), or restrict attention to situations where the gap between γ_P and γ_V is sufficiently large.

Finally, because of Blackwell's informativeness conditions, the variance and bias terms will on their own never generate prices that are more variable than realized dividends. This form of excess price variability results only if there is enough over-reaction to over-turn the Variance and Bias terms, which would be on their own smaller than the variability in realized dividends.

4 Applications

In this section, we apply our theory to reconsider the Modigliani-Miller Theorem, the excess volatility puzzle, and the impact of public disclosures and transparency on asset prices.

4.1 Splitting Cash-flows to influence market value

The Modigliani-Miller theorem states that in perfect and complete financial markets, splitting a Cash-flow into two different securities, and selling these claims separately to investors does not influence its total market value (Modigliani and Miller, 1958). Here we show that with noisy information aggregation, the Modigliani-Miller theorem remains valid only if the different claims are sold to investor pools with *identical* informational characteristics. When the investor pools for different claims have different characteristics, then the nature of the split influences the seller's revenue. The seller in turn can increase her revenues by tailoring the split to the different investor types.

Consider a seller who owns claims on a stochastic dividend $\pi(\cdot)$. This Cash flow is divided into two parts, π_1 and π_2 , both monotone in θ , such that $\pi_1 + \pi_2 = \pi$, and then sold to traders in two separate markets. We suppose that either π_1 is dominated by downside risks, or π_2 is dominated by upside risks (or both). For each claim, there is a unit measure of informed traders, who obtain a noisy private signal $x_i \sim \mathcal{N}(\theta, \beta_i^{-1})$, and a noise trader demand $\Phi(s_i)$, where

$$\begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = \mathcal{N} \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_{u,1}^2 & \rho\sigma_{u,1}\sigma_{u,2} \\ \rho\sigma_{u,1}\sigma_{u,2} & \sigma_{u,2}^2 \end{pmatrix} \right)$$

That is, each market is affected by a noise trader shock s_i with market-specific noise parameter $\sigma_{u,i}^2$. The environment is then characterized by the market-characteristics β_i and $\sigma_{u,i}^2$, and by the

correlation of demand shocks across markets, ρ . Traders are active only in their respective market. However, we consider both the possibility that traders observe and condition on prices in the other market (informational linkages), and the possibility that they don't (informational segregation).

Under informational segregation, the analysis of the two markets can be completely separated; any correlation between the two in prices is the result of correlation in demand shocks, as well as the common underlying fundamental, but this doesn't influence expected revenues. The equilibrium characterization from proposition 1 applies separately in each market:

$$P_i(z_i) = \mathbb{E}(\pi_i(\theta)|x = z_i, z_i; \beta_i, \sigma_{u,i}^2) \text{ and } V_i(z_i) = \mathbb{E}(\pi_i(\theta)|z_i; \beta_i, \sigma_{u,i}^2).$$

The seller's total expected revenue in excess of the cash flow's expected dividend value is then given by $W_{\pi_1}(\sigma_{P,1}) + W_{\pi_2}(\sigma_{P,2})$, where $\sigma_{P,i}$ is determined as in lemma 2, and denotes the level of informational frictions in each market.

With informational linkages, the equilibrium analysis has to be adjusted to incorporate the information contained in price 1 for the traders in market 2, and vice versa. The characterization proceeds along the same lines as the previous model. Since expected dividends are monotone, informed traders in market i will buy a security if and only if their private signal exceeds a threshold $\hat{x}_i(\cdot)$, where $\hat{x}_i(\cdot)$ is conditioned on both prices. By market-clearing, it must be the case that $\hat{x}_i(\cdot) = z_i \equiv \theta + 1/\sqrt{\beta_i} \cdot s_i$. Observing P_i is then isomorphic to observing z_i , and observing both prices is isomorphic to observing (z_1, z_2) . We let (z_1, z_2) denote the state, and consider equilibrium price functions $P_1(\cdot)$ and $P_2(\cdot)$ that are measurable w.r.t. (z_1, z_2) . It is then straight-forward to characterize posterior beliefs over θ using Bayes'rule, and to characterize the traders' indifference conditions and hence the market price functions, and expected dividends, conditional on (z_1, z_2) :

$$\begin{aligned} P_1(z_1, z_2) &= \mathbb{E}(\pi_1(\theta)|x = z_1; z_1, z_2) \text{ and } V_1(z_1, z_2) = \mathbb{E}(\pi_1(\theta)|z_1, z_2) \\ P_2(z_1, z_2) &= \mathbb{E}(\pi_2(\theta)|x = z_2; z_1, z_2) \text{ and } V_2(z_1, z_2) = \mathbb{E}(\pi_2(\theta)|z_1, z_2) \end{aligned}$$

In Appendix B, we fully characterize expected prices and dividends for this two asset model. In particular, we show the following modified version of lemma 2.

Lemma 3 (Unconditional Wedge with two assets) *For each cash-flow π_i , the unconditional information aggregation wedge W_{π_i} is characterized by*

$$\begin{aligned} W_{\pi_i}(\sigma_{P,i}) &= \int_0^\infty (\pi'_i(\theta) - \pi'_i(-\theta)) \left(\Phi\left(\frac{\theta}{\sigma_\theta}\right) - \Phi\left(\frac{\theta}{\sigma_{P,i}}\right) \right) d\theta, \text{ where} \\ \sigma_{P,i}^2 &= \sigma_\theta^2 + \frac{\beta_i}{(\beta_i + V)^2} (1 + \sigma_{u,i}^2) \text{ and } V = \sigma_\theta^{-2} + \frac{1}{1 - \rho^2} \left(\frac{\beta_1}{\sigma_{u,1}^2} + \frac{\beta_2}{\sigma_{u,2}^2} - 2\rho \frac{\sqrt{\beta_1\beta_2}}{\sigma_{u,1}\sigma_{u,2}} \right) \end{aligned}$$

Therefore, the cases of informational segregation and informational linkages therefore only differ in terms of how our measure of informational frictions $\sigma_{P,i}$ depends on the underlying primitive parameters in each case, but given for given values of π_i and $\sigma_{P,i}$, the seller's expected revenue net of expected dividends in both cases is $W_{\pi_1}(\sigma_{P,1}) + W_{\pi_2}(\sigma_{P,2})$. With this, we can state a first version of the Modigliani-Miller theorem, for expected revenues, in our model.

Proposition 2 (Modigliani-Miller I) *(i) The cash-flow split does not affect the seller's expected revenue, if and only if the market characteristics are identical: $\sigma_{P,1} = \sigma_{P,2}$.*

(ii) The seller's expected revenues exceed expected cash-flows whenever $\sigma_{P,1} > \sigma_{P,2}$, and are less than expected cash-flows whenever $\sigma_{P,1} < \sigma_{P,2}$.

The key to this proposition is that, for given values of σ_P , the expected information aggregation wedge is additive across Cash flows: $W_{\pi_1}(\sigma_P) + W_{\pi_2}(\sigma_P) = W_{\pi_1+\pi_2}(\sigma_P)$, for any σ_P , π_1 and π_2 . If the two markets have identical characteristics, i.e. $\sigma_{P,1} = \sigma_{P,2}$, only the combined cash flow matters for the total wedge - i.e. the Modigliani-Miller result applies. If on the other hand the two markets have different informational characteristics, then the additivity of the wedge for a given value of σ_P implies that the seller's total expected revenue is equal to

$$\begin{aligned} W_{\pi_1}(\sigma_{P,1}) + W_{\pi_2}(\sigma_{P,2}) &= W_{\pi}(\sigma_{P,2}) + W_{\pi_1}(\sigma_{P,1}) - W_{\pi_1}(\sigma_{P,2}), \text{ or} \\ W_{\pi_1}(\sigma_{P,1}) + W_{\pi_2}(\sigma_{P,2}) &= W_{\pi}(\sigma_{P,1}) + W_{\pi_2}(\sigma_{P,2}) - W_{\pi_2}(\sigma_{P,1}). \end{aligned}$$

This expression decomposes the total revenue net of expected dividends into a wedge associated with the total cash flow π , plus a difference term that depends on the cash flow split, i.e. π_1 or π_2 . Depending on the comparison between $\sigma_{P,1}$ and $\sigma_{P,2}$, and the upside or downside risk in π_1 or π_2 , this difference is either positive or negative.

Consider now how a seller can influence expected revenues by assigning the Cash flows π_1 and π_2 to two pools of investors, which are characterized by $\sigma_{P,1}$ and $\sigma_{P,2}$. Assume without loss of generality that $\sigma_{P,2} > \sigma_{P,1}$. As an immediate corollary of the previous proposition, the seller's revenues are maximized, whenever π_1 is sold to the investor pool characterized by $\sigma_{P,1}$, and π_2 is sold to the pool that is characterized by $\sigma_{P,2}$. Intuitively, the seller exploits the information aggregation wedge to manipulate revenues by matching the pool of investors that is characterized by bigger informational frictions with the upside risk, so as to maximize the gains from the positive wedge resulting on the upside, while selling the downside risks to an investor pool with smaller informational frictions, so as to reduce the losses incurred on the negative wedge on the downside. This logic is pushed further by the next proposition, which considers how the seller can exploit the heterogeneity in investor pools if she gets to design the split of π into π_1 and π_2 .

Proposition 3 (Designing Cash flows) *The seller maximizes her expected revenues by splitting cash flows according to $\pi_1^*(\theta) = \min\{\pi(\theta), \pi(0)\}$ and $\pi_2^*(\theta) = \max\{\pi(\theta) - \pi(0), 0\}$, and then assigning π_1^* to the investor pool with the lower value of σ_P .*

Figure 3: Optimal cash-flow design

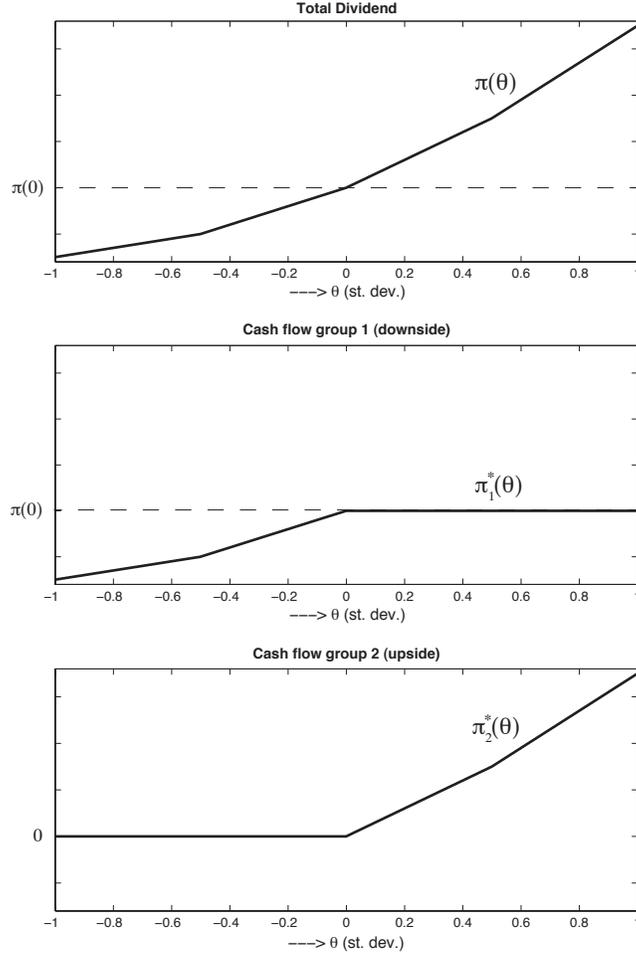


Figure 3 sketches the optimal dividend split for an arbitrary dividend function. The seller maximizes the total proceeds by assigning all the cash flow below the straight line corresponding to $\pi(0)$ (white area) to the investor group with the lowest information friction parameter; $\sigma_{P,1}$, and the complement (gray area) to the investor group with the highest friction; $\sigma_{P,2}$. It is easy to show that any other arbitrary division of cash flows $\{\pi_1(\cdot), \pi_2(\cdot)\}$ implies that both $\pi_1^{*'}(\theta) - \pi_1^{*'}(-\theta) \leq \pi_1'(\theta) - \pi_1'(-\theta)$, and $\pi_2^{*'}(\theta) - \pi_2^{*'}(-\theta) \geq \pi_2'(\theta) - \pi_2'(-\theta)$; i.e. π_1 has less downside risk than π_1^* , and π_2 has less upside risk than π_2^* . The increasing difference property of part iv) in Theorem 1

then implies that any transfer of cash flows between investor groups resulting from the alternative split reduces the total proceeds of the issuance. Intuitively, the optimal split loads the entire downside risk on the investor group that discounts the price of the claim the least with respect to its expected payoff (because of the low friction parameter; $\sigma_{P,1}$), while loading the entire upside risk to the group that overvalues the claim the most with respect to its expected dividend (due to the high information frictions; $\sigma_{P,2}$). When $\pi(\cdot) > 0$, this split has a straight-forward interpretation in terms of debt and equity, with a default point on debt that is set at the prior mean $\theta = 0$.

An important limitation of the discussion in this section is that here we are taking as given the differences in market characteristics. Moreover, we are implicitly assuming that, given these differences, the seller can freely assign the cash-flows to these two pools. In practice the situation is of course more complicated, because the investor's incentives to obtain information also depends on the asset risks they face. Analyzing this interplay between investor's information choices and the resulting market characteristics, along with the seller's security design question is clearly beyond the scope of this paper, but an important avenue for further work. The results here are mainly designed to highlight the possibility of systematic departures from Modigliani and Miller's (1958) irrelevance result, and show that limits to arbitrage give the owner of a cash flow distinct possibility to manipulate its market value through strategic security design.

We conclude this section by stating a second version of the Modigliani-Miller theorem for realized revenues, $P_{\pi_1}(z_1, z_2) + P_{\pi_2}(z_1, z_2)$. The original Modigliani-Miller theorem holds also at an interim stage conditional on new information. Here, the interim version requires that for each realization (z_1, z_2) , the marginal traders in the two markets hold identical beliefs.

Proposition 4 (Modigliani-Miller II) (i) *With information segregation, $P_{\pi_1}(z_1, z_2) + P_{\pi_2}(z_1, z_2) = P_{\pi}(z_1, z_2)$ for all (z_1, z_2) , if and only if the signals are perfectly correlated ($\rho = 1$), and the two market prices have the same informational parameters: $\beta_1 = \beta_2$ and $\sigma_{u,1}^2 = \sigma_{u,2}^2$.*

(ii) *With informational linkages, $P_{\pi_1}(z_1, z_2) + P_{\pi_2}(z_1, z_2) = P_{\pi}(z_1, z_2)$ for all (z_1, z_2) , if and only if the signals are perfectly correlated ($\rho = 1$), and either $\beta_1 = \beta_2$ and $\sigma_{u,1}^2 = \sigma_{u,2}^2$ or $\beta_1\sigma_{u,1}^{-2} \neq \beta_2\sigma_{u,2}^{-2}$.*

The perfect correlation reduces the noise to a single common shock. That this is necessary for the theorem to hold under segregation is immediate. It is also necessary for the case with informational linkages, because of the different weighting between the signals for the marginal traders in the two markets. In addition the signal distributions need to be the same, requiring that $\beta_1\sigma_{u,1}^{-2} = \beta_2\sigma_{u,2}^{-2}$. Finally, the wedge needs to be the same in the two markets, or $\sigma_{P,1} = \sigma_{P,2}$.

Together these conditions imply that the two markets have identical informational characteristics, and the marginal trader therefore holds identical beliefs. In the case with informational linkages, we need to consider one additional possibility, which is that when $\beta_1\sigma_{u,1}^{-2} \neq \beta_2\sigma_{u,2}^{-2}$ and $\rho = 1$, the observations of two signals enables every trader to perfectly infer θ and u from the two prices, regardless of the informational parameters.

An interim version of the theorem therefore requires perfect correlation in the noise in different markets, on top of identical informational characteristics.

4.2 Dynamic Trading

Here we consider a simple dynamic extension of our asset market model. The main insight we recover from the dynamic extension is that the anticipation of positive expected wedges in future periods raises asset prices in the present. Hence, we show that a long-lived asset with convex payoff function and a lower bound on per period dividends can in fact be priced above its fundamental value in all periods, and all states.

Time is discrete and infinite, and in each period a new trading round takes place, with informed traders and noise traders. As before the total asset supply is 1. The asset pays dividends $\pi(\theta_t)$ after the current trading round has taken place (hence θ_t is publicly known before the start of period $t + 1$). For simplicity we assume that the fundamental θ_t and the stochastic demand shock u_t are distributed as specified as in section 2, and iid over time.⁹ Traders are long-lived and risk-neutral and discount the future at a rate $\delta \in (0, 1)$.

In this case, since there is no persistence in the process, trading in round t only aggregates information about the current fundamental, but includes the anticipation of future prices. Formally, the payoff to a share bought in period t is $\pi(\theta_t) + \delta P(z_{t+1})$, where $P(z_{t+1})$ is the price in period $t + 1$, contingent on the period $t + 1$ state z_{t+1} . The price then satisfies the following recursive characterization:

$$P_\pi(z_t) = \mathbb{E}(\pi(\theta_t) + \delta P_\pi(z_{t+1}) | x = z_t, z_t)$$

where as the expected dividend value of the asset satisfies

$$V_\pi(z_t) = \mathbb{E}(\pi(\theta_t) + \delta V_\pi(z_{t+1}) | z_t)$$

Using the fact that in the iid case, the expected future prices and dividend values correspond to the

⁹It is possible but outside the scope of the current paper to extend the analysis to allow for persistent fundamental processes. The i.i.d. case is sufficient to convey the core insights that anticipated future wedges influence the current level of the wedge.

unconditional ones, we have the following characterization of the information aggregation wedge in the dynamic model:

$$W_\pi(z_t) = w_\pi(z_t) + \delta \mathbb{E}(W_\pi(z)) = w_\pi(z_t) + \frac{\delta}{1-\delta} \mathbb{E}(w_\pi(z)),$$

where $w_\pi(z_t) = \mathbb{E}(\pi(\theta_t) | x = z_t, z_t) - \mathbb{E}(\pi(\theta_t) | z_t)$

is the wedge resulting from the current period payoffs, and $\mathbb{E}(w_\pi(z))$ the corresponding unconditional expectation. Thus, the information aggregation wedge in the dynamic setting depends on both the wedge resulting from current payoffs, and the expected discounted future wedge. Even when the concurrent wedge is negative (at low realizations of z), the overall wedge may still be positive because traders anticipate higher share prices in the future. The following proposition formalizes this observation.

Proposition 5 (Dynamic Wedge) *Suppose that $\pi(\theta)$ is bounded below, increasing, and convex. Then, for any $\sigma_P > \sigma_\theta$, there exists $\hat{\delta} < 1$ such that for all $\delta > \hat{\delta}$, $W(z) > 0$, for all z .*

Thus, for claims that have a lower bound on payoffs (for example, requiring them to be non-negative), and that generate a positive unconditional wedge, the market price can exceed expected dividends at all times and in all states of the world. Moreover, as shown by Theorem 2, $w_\pi(z)$ and hence $W_\pi(z)$ and $P_\pi(z)$ may be arbitrarily volatile if σ_P is sufficiently high. This may occur for instance with the exponential payoffs in example 2, where high expected levels and high volatility are jointly sustained (for given dividend volatility), whenever σ_P is sufficiently large. Symmetrically, a claim whose payoffs are bounded above may be undervalued in all future states.

To summarize, a simple repeated version of our model implies that the future anticipated wedges (positive or negative) can lead to higher or lower current prices. On levered claims, it can induce permanent over-valuation and high price volatility relative to fundamentals. Overvaluation and price volatility can both become arbitrarily large, when the market price is sufficiently noisy.

Our results can shed a new perspective into the well documented “excess volatility puzzle” (Le Roy and Porter 1981; Shiller 1981). The literature consensus is that excess volatility arises from variation in the stochastic discount factor, tightly linked to risk aversion in economies that allow a representative agent characterization (Campbell and Shiller 1988; Cochrane, 1992). Such conclusion follows from the fact that in representative Bayesian agent models, the volatility of expected dividends can never exceed realized dividends (West, 1988), whose importance in variance decomposition tests is very small.¹⁰

¹⁰For a recent digression, see Chen and Zhao (2009).

In contrast, our theory suggests that high return volatility could result from volatile dividend expectations in a Bayesian environment, despite low variation in observed dividends, as long as the informational frictions stressed above are severe enough. In our simple dynamic example with a fixed discount factor, return volatility arises from the interplay between noise trading shocks and dispersed information. When noise trading is highly volatile, market information in prices is noisy and traders beliefs remain heterogeneous in equilibrium. With finite precision of private signals, large shifts in noisy demand are absorbed by large shifts in the identity of the marginal trader, resulting in high price volatility. Moreover, the ratio between price and realized dividend volatility can be made arbitrarily large by increasing the variance of noise trading shocks.

Whether heterogeneous expectations can account for excess price volatility is an empirical question we do not address here. Rather, the contribution of our model in this respect is to offer a theoretical framework, fully consistent with agent rationality, where this channel is not ruled out by the mere observation that the variability of actual dividends is modest.

4.3 Public Disclosures

Here, we consider an extension that adds an exogenous public signal to the model. When all information among investors is common, the asset price accurately reflect current expectations of dividends, i.e. $P(\cdot) = V(\cdot)$. In this case, the logic of Blackwell's theorem implies that the provision of more accurate public signals always brings prices closer to realized dividends, reducing pricing errors. This also reflects itself in higher price variability (but always less than variability of dividends), and a higher correlation between prices and expected dividends.

Here, we show that these implications no longer apply, at least for sufficiently noisy public signals, once information aggregation is noisy. In our model, pricing errors occur both because of the noise in the market information, and because of the information aggregation wedge. Surprisingly, the effect of increased transparency on the information aggregation wedge and the pricing errors turns out to be ambiguous (and globally, non-monotone): on the one hand, such information reduces the importance of the market signal, therefore (when it is sufficiently precise) eliminates both sources of pricing errors. On the other hand, the public signal introduces a second source of systematic mispricing: The flipside to the over-reaction to market fundamentals is that the price reacts too little to exogenous public news. When the public signal is sufficiently noisy, this can drive prices further away from expected dividends and increase the magnitude of the wedge, and the pricing error. At the same time, when information frictions are sufficiently severe so as to generate very high price volatility, public announcements also stabilize the variability of prices by crowding

out the market signal from the marginal trader's expectations. In summary, the conventional logic of how public disclosures affect share prices gets over-turned for disclosures that are sufficiently noisy.

To formalize these insights, consider the same model as in section 2, but suppose that in addition all traders observe a noisy public signal y , which is distributed according to $y \sim \mathcal{N}(\theta, \alpha^{-1})$, and independent of the noise in the asset supply. We can interpret y as a public announcement about the firm's fundamentals. Here we discuss the implications of such announcements for the asset price volatility and the information aggregation wedge. Following the same steps as in section 2, it follows immediately that the equilibrium price and the expected dividend value are now functions of y and z , and take the form

$$P(y, z) = \mathbb{E}(\pi(\theta)|y, x = z, z) = \int \pi(\theta) d\Phi \left(\sqrt{\sigma_\theta^{-2} + \alpha + \beta + \beta\sigma_u^{-2}} \left(\theta - \frac{\alpha y + (\beta + \beta\sigma_u^{-2})z}{\sigma_\theta^{-2} + \alpha + \beta + \beta\sigma_u^{-2}} \right) \right)$$

$$V(y, z) = \mathbb{E}(\pi(\theta)|y, z) = \int \pi(\theta) d\Phi \left(\sqrt{\sigma_\theta^{-2} + \alpha + \beta\sigma_u^{-2}} \left(\theta - \frac{\alpha y + \beta\sigma_u^{-2}z}{\sigma_\theta^{-2} + \alpha + \beta\sigma_u^{-2}} \right) \right)$$

One immediately observes that the information aggregation wedge disappears, when $\alpha \rightarrow \infty$, i.e. when the public signal becomes infinitely precise. In this limit, $V(y, z)$ and $P(y, z)$ both converge to $\pi(y)$, as y completely crowds out z from the expectations. We effectively approach a common information limit, in which private information no longer plays any role for equilibrium prices, and the information aggregation wedge disappears.¹¹

For finite levels of noise, the effects of disclosure on prices, expected dividend values and the wedge depend on the informativeness of the private signals and the market price. It is always the case that a public disclosure increases the ex ante variability of $V(y, z)$, because the provision of more precise information makes the posterior more uncertain from an ex ante perspective. However contrary to the effects of a highly precise public signal, a noisy disclosure may increase the size of the information aggregation wedge, if the market is sufficiently informative on its own, i.e. $\beta\sigma_u^{-2}$ is high relative to σ_θ^{-2} . Moreover, a public disclosure (even if it is noisy) may reduce price variability, if the market signal is noisy and the initial information aggregation wedge is large, i.e. the price is much more variable than the expected dividend value.

To illustrate these points and obtain sharper results of the effects of public disclosures with

¹¹If the market and private information are allowed to increase in proportion with α , a wedge remains in the limit for given (y, z) , whenever $y \neq z$, but then $y - z$ converges to zero in probability. More generally, the wedge disappears, whenever $\alpha + \beta\sigma_u^{-2} \rightarrow \infty$, i.e. whenever the publicly available information becomes infinitely precise in the limit.

imperfect precision, we focus on the linear case, $\pi(\theta) = \theta$.¹² Then,

$$P(y, z) = \frac{\alpha y + (\beta + \beta\sigma_u^{-2})z}{\sigma_\theta^{-2} + \alpha + \beta + \beta\sigma_u^{-2}}, \text{ and } V(y, z) = \frac{\alpha y + \beta\sigma_u^{-2}z}{\sigma_\theta^{-2} + \alpha + \beta\sigma_u^{-2}}$$

$$W(y, z) = \left(\frac{\beta + \beta\sigma_u^{-2}}{\sigma_\theta^{-2} + \alpha + \beta + \beta\sigma_u^{-2}} - \frac{\beta\sigma_u^{-2}}{\sigma_\theta^{-2} + \alpha + \beta\sigma_u^{-2}} \right) \left(z - \frac{\alpha}{\sigma_\theta^{-2} + \alpha} y \right)$$

and the prior variances of the price, the expected dividend value and the wedge are

$$\text{Var}(V(y, z)) = \frac{\alpha + \beta\sigma_u^{-2}}{\sigma_\theta^{-2} + \alpha + \beta\sigma_u^{-2}} \frac{1}{\sigma_\theta^{-2}}$$

$$\text{Var}(P(y, z)) = \frac{\alpha + \beta + \beta\sigma_u^{-2}}{\sigma_\theta^{-2} + \alpha + \beta + \beta\sigma_u^{-2}} \frac{1}{\sigma_\theta^{-2}} + \frac{\beta(1 + \sigma_u^{-2})/\sigma_u^{-2}}{(\sigma_\theta^{-2} + \alpha + \beta + \beta\sigma_u^{-2})^2}$$

$$\text{Var}(W(y, z)) = \left(\frac{\beta + \beta\sigma_u^{-2}}{\sigma_\theta^{-2} + \alpha + \beta + \beta\sigma_u^{-2}} - \frac{\beta\sigma_u^{-2}}{\sigma_\theta^{-2} + \alpha + \beta\sigma_u^{-2}} \right)^2 \left(\frac{1}{\beta\sigma_u^{-2}} + \frac{1}{\sigma_\theta^{-2} + \alpha} \right)$$

We then have the following proposition:

Proposition 6 (Public Disclosures) *Comparative Statics w.r.t. the precision of public disclosures:*

(i) $\text{Var}(V(y, z))$ is increasing in α .

(ii) If $\sigma_u^{-2} \geq 2$, $\text{Var}(P(y, z))$ is increasing in α . Otherwise $\text{Var}(P(y, z))$ is increasing in α if and only if $\alpha \geq \alpha_P$ (and decreasing otherwise), where $\alpha_P > 0$ is implicitly defined as

$$\frac{\sigma_\theta^{-2} + \alpha_P}{\sigma_\theta^{-2} + \alpha_P + \beta + \beta\sigma_u^{-2}} = 1 - \frac{\sigma_u^{-2}}{2}$$

(iii) $\text{Var}(W(y, z))$ is decreasing in α , if and only if $\alpha \geq \alpha_W$ (and increasing otherwise), where α_W is implicitly defined as

$$\frac{\sigma_\theta^{-2} + \alpha_W}{\sigma_\theta^{-2} + \alpha_W + \beta + \beta\sigma_u^{-2}} = \frac{1}{2} \frac{\beta\sigma_u^{-2}}{\sigma_\theta^{-2} + \alpha_W + \beta\sigma_u^{-2}}$$

The first part is standard and follows from the information value of the disclosure. The comparative statics for $P(y, z)$ and $W(y, z)$ on the other hand are ambiguous. Starting from low levels of precision α , the public disclosure will first crowd the market signal out from the expected dividend value, and then from the price. If the market price is already very informative ($\beta\sigma_u^{-2}$ is high relative to σ_θ^{-2}), then a disclosure that contains information about θ reduces the impact of z on $V(y, z)$ more than the impact of z on $P(y, z)$. If $\sigma_\theta^{-2} + \alpha$ is sufficiently low relative to β and $\beta\sigma_u^{-2}$, this

¹²The same results as for the linear case can also be established for the CARA-normal case, by appropriately adjusting the informativeness of the market signal, along the lines of the analysis in the next section.

increases the magnitude of the wedge. On the other hand, once $\sigma_\theta^{-2} + \alpha$ is sufficiently high, it reduces the impact of z on both $P(y, z)$ and $V(y, z)$ and the wedge therefore becomes smaller.

Moreover, if the market price is very noisy, relative to the private signals, the public announcement also reduces the variability of prices by mitigating the over-reaction of prices to z . This is not inconsistent with the wedge becoming larger, because at the same time the prices and expected dividends become less correlated, because of their differential responses to y and z . Only once α is sufficiently large, the price volatility increases again in α , as the public signal brings the price closer in line with the fundamental value.

To summarize, the information aggregation wedge introduces some subtle effects into the effect of public disclosures on asset prices, which over-turn the standard logic on effects of disclosures that is derived from Blackwell's theorem. The effects are similar to those discussed by Morris and Shin (2002) in the context of a beauty contest game, or Allen, Morris and Shin (200?) in a dynamic asset market with short-lived traders, where excess volatility in actions resulted from the players' over-reaction to a public signal. Here, the over-reaction is to endogenous market information, and the public disclosure shifts the extent of over-reaction in prices relative to expected dividends. A noisy public signal reduces the impact of z on prices and this can stabilize prices, especially when the price is very volatile and its informativeness is low. At the same time, such a signal may actually increase the over-reaction of prices to z , because the disclosure at first affects the response of prices to z by less than it affects expected dividends. This makes the information aggregation wedge larger.

By confining itself to a static model, the present analysis does not directly speak to the large literature on asset pricing anomalies, such as momentum and reversal in price dynamics,¹³ or the literature on the post-earnings announcement drift and under-reaction to public disclosures on impact.¹⁴ However, by offering both channels for under-reaction to external news on impact, as well as overreaction to endogenous market signals, our model offers a rich set of elements that may be useful to speak to these empirical issues in a proper dynamic extension of the current framework.

5 Discussion

In this section, we explore the robustness of the information aggregation wedge to changes in the model's core assumptions. We show that the information aggregation wedge is (i) robust to alternative specifications of the prior distribution of θ , as well as the distribution of private signals,

¹³Daniel, Hirshleifer and Subrahmanyam, 1998; Hong and Stein, 1999; Vayanos and Wooley, 2008.

¹⁴Bernard and Thomas, 1985; Jegadeesh and Titman, 1993.

(ii) inversely related to the extent of arbitrage activity by risk-neutral, uninformed traders (or more specifically the demand elasticity of noise and uninformed traders), and (iii) robust to allowing for risk aversion in the specification of the canonical CARA-normal model.

5.1 Distributional Assumptions

We first show that the features of the information aggregation wedge, which we identified in the previous section, are not an artefact of some of the more special and restrictive assumptions that we imposed. In fact, these assumptions mainly served to retain the tractability of Bayesian updating with normal distributions, which offered more concrete interpretations of the comparative statics emerging from the wedge, but they did not play a fundamental role in its presence.

Specifically, suppose as before that traders are risk-neutral and face position limits at 0 and 1, but consider now the following generalization: (i) θ is distributed according to an arbitrary smooth prior on \mathbb{R} , (ii) private signals are distributed iid according to a distribution with cdf $F(\cdot|\theta)$, which satisfies the monotone likelihood ratio property, and (iii) the noise trader demand D is distributed according to an arbitrary smooth distribution with cdf $G(\cdot)$ on $[0, 1]$.¹⁵ This formulation imposes few restrictions on the distributions apart from the monotone likelihood ratio property for private signals (to insure monotonicity of posteriors and trading behavior), and smoothness assumptions to maintain continuity and invertibility of the price function. The following equilibrium characterization is then a direct generalization of proposition 1:

Proposition 7 (Distributional Assumptions) *In the unique asset market equilibrium, the price function $P(z)$ is characterized by*

$$P(z) = \mathbb{E}(\pi(\theta)|x = z, z) = \frac{\int \pi(\theta)h(\theta)g(F(z|\theta))f^2(z|\theta)d\theta}{\int h(\theta)g(F(z|\theta))f^2(z|\theta)d\theta},$$

and the traders' threshold function is $\hat{x}(p) = z = P^{-1}(p)$. The expected dividend conditional on z , on the other hand takes the form

$$V(z) = \mathbb{E}(\pi(\theta)|z) = \frac{\int \pi(\theta)h(\theta)g(F(z|\theta))f(z|\theta)d\theta}{\int h(\theta)g(F(z|\theta))f(z|\theta)d\theta}.$$

Risk-neutrality, position limits and the MLRP on private signals allow us to retain a threshold characterization for the buying strategy of informed traders. We can then use the market-clearing condition to re-define the information conveyed through P by an observable state variable $z = \hat{x}(P)$, whose realization depends only on the exogenous shocks θ and D . The indifference condition for the

¹⁵Informed trader positions are restricted to $[0, 1]$, so the market cannot clear if $D > 1$ or $D < 0$.

marginal trader implies that $P(z) = \mathbb{E}(\pi(\theta)|x = z, z)$, that is, the asset price equal the expectation of an informed trader with public signal z and an independent private signal x whose realization also equals z , while the expected dividend conditions on z only once, as the market signal. The final step in the proof consists in characterizing the distribution of z from the primitive distributions, and showing that its conditional pdf is $\varphi(z|\theta) = g(F(z|\theta))f(z|\theta)$.

This characterization shows that the logic of the wedge is not directly tied to the specific distributional assumptions that we have made (other than the MLRP assumption on signals). This still leaves as an open question to what extent the results of Theorem 1 and 2 make use of this additional structure of normality or symmetry in the signals - an issue that is relevant for the economic conditions and interpretation underlying these two theorems. We hope to address this question in future work.

5.2 Price impact of information

Next, we generalize our previous formulation to allow for a response of uninformed traders to perceived excess returns on the asset, as well as stochastic trading motives which are unrelated to dividend expectations (for example, liquidity or hedging needs). We keep the same model as in section 2, but consider the following formulation for asset demand:

$$D(u, P) = \Phi(u + \eta(\mathbb{E}(\pi(\theta)|P) - P)), \quad (8)$$

with $u \sim \mathcal{N}(0, \sigma_u^2)$. Uninformed traders' demand is increasing in the expected return conditional on the price, $\mathbb{E}(\pi(\theta)|P) - P$, with an elasticity given by η .¹⁶ The parameter η captures the responsiveness of uninformed traders to the expectation of dividends in excess of prices, or in other words, the extent to which they are willing or able to arbitrage away the difference between expected price and dividend value. Equivalently, η measures the price impact of private information which relates naturally to the concept of market liquidity.

We follow our previous equilibrium characterization and asset prices with minor changes to account for the endogeneity of demand to asset prices. Market-clearing implies $\Phi(\sqrt{\beta}(\hat{x}(P) - \theta)) = \Phi(u + \eta(\mathbb{E}(\pi(\theta)|P) - P))$, or

$$z = \theta + \frac{1}{\sqrt{\beta}}u = \hat{x}(P) - \eta/\sqrt{\beta} \cdot (\mathbb{E}(\pi(\theta)|P) - P).$$

Observing P is thus isomorphic to observing $z \sim \mathcal{N}(\theta, \sigma_u^{-2}/\beta)$, and Lemma 1 continues to hold without any changes. Using the fact that the expected dividend is $\mathbb{E}(\pi(\theta)|P) = \mathbb{E}(\pi(\theta)|z) =$

¹⁶Exactly the same analysis can be conducted if uninformed traders responded to the expected return $\mathbb{E}(\pi(\theta)|P)/P$ (provided the latter is well-defined, i.e. $\pi(\theta)$ is always non-negative), instead of the payoff difference $\mathbb{E}(\pi(\theta)|P) - P$.

$V(z)$, the equilibrium price function is implicitly defined by the marginal trader's indifference condition

$$P(z) = \mathbb{E}(\pi(\theta) | \hat{x}(P), z) = \mathbb{E}\left(\pi(\theta) | z + \eta/\sqrt{\beta} \cdot (V(z) - P(z)), z\right).$$

This condition implicitly defines the equilibrium price. Let $P(z; \eta)$ denote the equilibrium price as a function of the elasticity parameter η , and $P(z; 0) = P(z)$ the price function with inelastic supply. The next proposition shows that the magnitude of the information aggregation wedge is inversely related to the uninformed trader's demand elasticity.

Proposition 8 (Price-elastic demand) *If $P(z; 0) = V(z)$, then $P(z; \eta) = V(z)$, for all η . If $P(z; 0) \neq V(z)$, then $|P(z; \eta) - V(z)|$ is strictly decreasing in η and $\lim_{\eta \rightarrow \infty} |P(z; \eta) - V(z)| = 0$.*

Therefore, the more elastically the uninformed and noise traders respond to the wedge between prices and expected dividends, the more they arbitrage away this difference, and the smaller the information aggregation wedge becomes. Fundamentally, the wedge results from the informed traders' impact on equilibrium prices. The more they move prices by acting on their private information, the larger is the wedge. In the inelastic case, the wedge was maximized as the informed traders fully determined prices, where their price impact vanishes in the limit as $\eta \rightarrow 0$. The parameter η can thus also be intuitively interpreted as a measure of the limits to arbitrage by uninformed, risk neutral outsiders.

We can illustrate these effects simply in the example with linear dividends: $\pi(\theta) = \theta$. In this case, the expected dividend value is $V(z) = \gamma_V \cdot z$, as before. The price however solves

$$P(z) = \gamma_P z + \frac{\eta\sqrt{\beta}}{\sigma_\theta^2(1 - \gamma_P)} (V(z) - P(z)) = \left(\gamma_P - (\gamma_P - \gamma_V) \frac{\eta\sqrt{\beta}}{\sigma_\theta^2(1 - \gamma_P) + \eta\sqrt{\beta}} \right) z.$$

Compared to the case with inelastic demand, the over-reaction is smaller, i.e. the coefficient in front of z is decreasing in η , and converges to 1 as $\eta \rightarrow \infty$. The wedge

$$W(z) = (\gamma_P - \gamma_V) \frac{\sigma_\theta^2(1 - \gamma_P)}{\sigma_\theta^2(1 - \gamma_P) + \eta\sqrt{\beta}} z$$

is also decreasing in η and vanishes as $\eta \rightarrow \infty$. The information aggregation wedge is therefore largest when uninformed traders are not actively arbitraging the expected return difference coming from the information aggregation wedge.

5.3 Information aggregation wedge in the CARA-normal model

Here we briefly compare our results with the pricing implications of a canonical model with CARA preferences and normally distributed signals and dividends.¹⁷ Consider the same market structure

¹⁷See, e.g., Hellwig (1980), Diamond and Verrecchia (1981), and Grossman and Stiglitz (1980).

as in section 2, but suppose that (i) traders start with initial share-holdings of zero, (ii) their preferences over terminal wealth are $u(w) = -\exp(-\gamma w)$, (iii) the dividend is normally distributed, $\pi(\theta) = \theta$, (iv) traders do not face limits on their portfolio holdings, and (v) the noise trader demand for shares is stochastic, normally distributed, according to $D \sim \mathcal{N}(\bar{D}, \sigma_u^2)$. All other elements of the model are as described before.

To highlight the role of the information aggregation wedge, we conjecture (and will verify) that in equilibrium, the information conveyed by the asset price is isomorphic to a normally distributed public signal $z(P) \sim \mathcal{N}(\theta, \tau_P^{-1})$. The informed traders' posterior conditional on $z(P)$ and their private signal x is normal, and their optimal demand satisfies

$$d(x, P) = \frac{\mathbb{E}(\theta|x, z(P)) - P}{\gamma \mathbb{V}(\theta|x, z(P))},$$

where $\mathbb{E}(\theta|x, z(P)) = \frac{\beta x + \tau_P z(P)}{\sigma_\theta^{-2} + \beta + \tau_P}$ and $\mathbb{V}(\theta|x, z(P)) = (\sigma_\theta^{-2} + \beta + \tau_P)^{-1}$.

Aggregating demand across traders and using the market-clearing condition

$$\int d(x, P) d\Phi(\sqrt{\beta}(x - \theta)) + D = 0,$$

the equilibrium price satisfies:

$$\begin{aligned} P &= \int \mathbb{E}(\theta|x, z(P)) d\Phi(\sqrt{\beta}(x - \theta)) + \frac{\gamma}{\sigma_\theta^{-2} + \beta + \tau_P} D \\ &= \frac{\beta \theta + \tau_P z(P)}{\sigma_\theta^{-2} + \beta + \tau_P} + \frac{\gamma}{\sigma_\theta^{-2} + \beta + \tau_P} D \end{aligned}$$

Therefore, observing P is isomorphic to observing $z(P) = \theta + \gamma/\beta \cdot (D - \bar{D})$, and $\tau_P = (\beta/\gamma)^2 \cdot \sigma_u^{-2}$.

The resulting equilibrium price function is

$$P(z) = \mathbb{E}(\theta|x = z, z) + \frac{\gamma \bar{D}}{\sigma_\theta^{-2} + \beta + \tau_P} = \frac{(\beta + \tau_P) z}{\sigma_\theta^{-2} + \beta + \tau_P} + \frac{\gamma \bar{D}}{\sigma_\theta^{-2} + \beta + \tau_P}.$$

On the other hand, the expected dividend, conditional on z is

$$\mathbb{E}(\theta|z) = \frac{\tau_P}{\sigma_\theta^{-2} + \tau_P} z.$$

Therefore, in the CARA-normal model, the difference between price and expected dividend, $P(z) - \mathbb{E}(\theta|z)$, can be decomposed into (i) a risk adjustment term $\gamma \bar{D} / (\sigma_\theta^{-2} + \beta + \tau_P)$, which compensates the traders for the risk they take on by holding the asset, and (ii) the information aggregation wedge $\mathbb{E}(\theta|x = z, z) - \mathbb{E}(\theta|z)$. The information aggregation wedge takes exactly the same form as in our benchmark model, and leads to prices that react more strongly to the state variable z than would

be justified by its information content about dividends. The sign of the risk adjustment term is determined by the sign of \bar{D} , i.e. the average position held by noise traders, and hence the negative of the informed traders' average position, and it scales with their degree of risk aversion γ and their posterior uncertainty $(\sigma_\theta^{-2} + \beta + \tau_P)^{-1}$. When $\bar{D} = 0$, i.e. the shares are on average in zero net supply, the risk adjustment term is zero, yet a systematic wedge between the price and expected dividends emerges because of the information aggregation wedge.

The CARA normal model thus has the same implications for conditional wedges as our risk-neutral model with position limits, suggesting that the information aggregation wedge is not unique to our model specification, and may in fact be a robust feature of models in which market prices aggregate dispersed private information. Remarkably, the only prior discussion of this observation that we are aware of is by Vives (2008), which briefly mentions the comparison between prices and expected dividends for the CARA case. The restrictions of the CARA model to normally distributed asset payoffs on the other hand limit the analysis to the symmetric case in which the ex ante expected wedge is zero. This precludes the results linking the shape of asset payoffs to their expected prices and dividends, and implies that any differences between unconditional expected prices and dividends are the consequence only of risk premia when the expected supply of shares differs from zero.

6 Concluding Remarks

We propose a model of asset prices in which heterogeneous investor information and limits to arbitrage generate a systematic wedge between the asset's price and its expected dividend value. This information wedge results from a compositional shift through which the informational content of prices is amplified, giving the market the appearance of attaching too much weight to the price signal, relative to its information content. This generates volatility of prices in excess of expected, and even realized dividends, even when traders are risk-neutral and discount future payoffs at a constant rate. It also generates price premia for securities that present upside risks, and price discounts for securities that present downside risks. The magnitude of these effects depend directly on the severity of the informational noise and the limits to arbitrage frictions.

We apply our theory to reconsider the Modigliani-Miller theorem, and show how a seller can affect revenues by an appropriate splitting of cash flows, selling upside risks in markets with bigger frictions, and downside risks in markets with less frictions. We also propose a dynamic extension of our model which jointly accounts for overvaluation and excess volatility, and we show that our

model can account for the under-reaction to public news, and generally speaking over-turns the logic of the impact of public disclosures that is derived from Blackwell's theorem.

We conclude with a few remarks on future research directions. Through our examples, we hope to have illustrated the tractability and versatility of our model for systematically analyzing asset prices and asset markets with heterogeneous information and limits to arbitrage. Our results also point to the potential of our theory to offer novel explanations for macro and micro asset pricing phenomena, such as excess volatility, impact of public announcements, and return predictability from prices, just to name a few. A first important avenue for future work is to consider whether our theory offers a plausible quantitative explanation for these and other asset pricing phenomena that appear a priori inconsistent with no arbitrage conditions or where risk-based explanations have reached their limits. This will require more systematic multi-period, and multi-asset extensions of our market model, both of which have already been touched upon in this paper in the context of specific examples.

A second important avenue for future work will be to merge our theory of limits to arbitrage with risk-based asset pricing models. The assumption of risk-neutrality, while clearly appealing for the purpose of identifying the role of limits to arbitrage for asset prices, and distinguishing it from risk-based models, is of course a major limitation in this direction. Our results from the CARA-normal model suggest that the main intuition and results survives in the symmetric normal case, and we see no reason to believe that this should be true for more general dividend functions, except that we lose the tractability of the CARA-normal-linear model.

A third important direction lies in the integration of financial market frictions with real decisions that give rise to the dividends. With few exceptions, much of the work in macroeconomics and corporate finance that focuses on these decisions tends to employ models of asset markets that is characterized by no-arbitrage conditions. On the normative side, one important practical conclusion drawn from these models is that asset prices can serve to alleviate incentive conflicts, incentivize managers and regulate firms or banks, by aligning the interests of those in charge of the decisions with those who suffer the consequences. A natural and important question to ask is then how these incentives should be regulated when asset markets can not be relied upon to align shareholder and manager incentives, and how the incentive schemes that rely on no arbitrage perform when there are limits to arbitrage in practice. Since our model offers a simple, tractable, and internally consistent theory of how limits to arbitrage drive prices away from fundamental values, we hope that it will prove to be a useful stepping stone for analyzing the effects of such asset mis-pricing

for real economic decisions and activity in more systematic ways.¹⁸

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¹⁸In a companion paper whose genesis preceeded the present work (Albagli, Hellwig, and Tsyvinski, 2011), we take a first stab at these questions by considering the interplay between information aggregation, firm decisions and managerial incentives in a simple model of informational feedback.

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7 Appendix A: Proofs

Proof of Lemma 1. Part (i): By market-clearing, $z = \hat{x}(P(z))$ and $\hat{x}(P(z')) = z'$, and therefore $z = z'$ if and only if $P(z) = P(z')$.

Part (ii): Since $P(z)$ is invertible, observing P is equivalent to observing $z = \hat{x}(P(z))$ in equilibrium. But $z|\theta \sim \mathcal{N}(\theta, \sigma_u^2/\beta)$, from which the characterization of $H(\cdot|x, P)$ follows immediately from Bayes' Law. ■

Proof of Proposition 1. Substituting $\hat{x}(P) = z$, a price function $P(z)$ is part of an equilibrium if and only if it satisfies (6) and is invertible. $\pi(\cdot)$ is strictly increasing, and an increase in z represents a first-order stochastic shift in the posterior over θ , so the price function $P_\pi(z)$ is continuous and monotone over its domain and spans its entire range, hence invertible. Moreover, all prices are observed in equilibrium (and hence out-of-equilibrium beliefs play no role). Thus, $P_\pi(z)$ defines the unique equilibrium in which prices are conditioned only on z . ■

Proof of Lemma 2. By the law of iterated expectations, $\mathbb{E}(V(z)) = \mathbb{E}(\pi(\theta)) = \int_{-\infty}^{\infty} \pi(\theta) d\Phi(\theta/\sigma_\theta)$. To find $\mathbb{E}(P(z))$, define $\sigma_P^2 = \sigma_\theta^2(1 + (\gamma_P/\gamma_V - 1)\gamma_P)$. Simple algebra shows that

$$\int_{-\infty}^{\infty} \sqrt{1 - \gamma_P\sigma_\theta} \phi\left(\frac{\theta - \gamma_P z}{\sqrt{1 - \gamma_P\sigma_\theta}}\right) d\Phi\left(\frac{\sqrt{\gamma_V}z}{\sigma_\theta}\right) = \frac{1}{\sigma_P} \phi\left(\frac{\theta}{\sigma_P}\right).$$

With this, we compute $\mathbb{E}(P(z))$:

$$\begin{aligned} \mathbb{E}(P(z)) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \pi(\theta) d\Phi\left(\frac{\theta - \gamma_P z}{\sqrt{1 - \gamma_P\sigma_\theta}}\right) d\Phi\left(\frac{\sqrt{\gamma_V}z}{\sigma_\theta}\right) \\ &= \int_{-\infty}^{\infty} \pi(\theta) \int_{-\infty}^{\infty} \sqrt{1 - \gamma_P\sigma_\theta} \phi\left(\frac{\theta - \gamma_P z}{\sqrt{1 - \gamma_P\sigma_\theta}}\right) d\Phi\left(\frac{\sqrt{\gamma_V}z}{\sigma_\theta}\right) d\theta \\ &= \int_{-\infty}^{\infty} \pi(\theta) \frac{1}{\sigma_P} \phi\left(\frac{\theta}{\sigma_P}\right) d\theta. \end{aligned}$$

Therefore, W_π is

$$\begin{aligned} W_\pi &= \int_{-\infty}^{\infty} \pi(\theta) \left(\frac{1}{\sigma_P} \phi\left(\frac{\theta}{\sigma_P}\right) - \frac{1}{\sigma_\theta} \phi\left(\frac{\theta}{\sigma_\theta}\right) \right) d\theta \\ &= \int_{-\infty}^{\infty} \pi'(\theta) \left(\Phi\left(\frac{\theta}{\sigma_\theta}\right) - \Phi\left(\frac{\theta}{\sigma_P}\right) \right) d\theta \\ &= \int_0^{\infty} (\pi'(\theta) - \pi'(-\theta)) \left(\Phi\left(\frac{\theta}{\sigma_\theta}\right) - \Phi\left(\frac{\theta}{\sigma_P}\right) \right) d\theta, \end{aligned}$$

where the first equality proceeds by integration by parts, the second by a change in variables, and the third step uses the symmetry of the normal distribution ($\Phi(-x) = 1 - \Phi(x)$). ■

Proof of Theorem 1. Parts (i) – (iii) follow immediately from lemma 2, the definition of upside and downside risk, and the fact that $\Phi(\theta/\sigma_\theta) > \Phi(\theta/\sigma_P)$ for all θ (since $\sigma_P > \sigma_\theta$). For part (iv) notice that

$$\begin{aligned} W_{\pi_2}(\sigma_P) - W_{\pi_1}(\sigma_P) &= \int_0^\infty \Delta(\theta) \left(\Phi\left(\frac{\theta}{\sigma_\theta}\right) - \Phi\left(\frac{\theta}{\sigma_P}\right) \right) d\theta, \\ \text{where } \Delta(\theta) &= \pi'_2(\theta) - \pi'_2(-\theta) - (\pi'_1(\theta) - \pi'_1(-\theta)). \end{aligned}$$

Since π_2 has less downside risk than π_1 , $\Delta(\theta) \geq 0$ for all θ , which implies that $W_{\pi_2}(\sigma_P) - W_{\pi_1}(\sigma_P)$ is increasing in σ_P . ■

Proof of Theorem 2. (i) The first inequality follows from standard arguments (Blackwell, 195?) for the value of information. For the second inequality, write $\mathbb{E}\left((P_\pi(z) - P_\pi(0))^2\right) - \mathbb{E}\left((V_\pi(z) - V_\pi(0))^2\right)$ as

$$\begin{aligned} &\mathbb{E}\left((P_\pi(z) - P_\pi(0))^2\right) - \mathbb{E}\left((V_\pi(z) - V_\pi(0))^2\right) \\ &= \mathbb{E}\left((P_\pi(z) - P_\pi(0))^2 \mid z \sim \mathcal{N}\left(0, \frac{\sigma_\theta^2}{\gamma_V}\right)\right) - \mathbb{E}\left((P_\pi(z) - P_\pi(0))^2 \mid z \sim \mathcal{N}\left(0, \frac{\sigma_\theta^2}{\gamma_P}\right)\right) \\ &\quad + \mathbb{E}\left((P_\pi(z) - P_\pi(0))^2 \mid z \sim \mathcal{N}\left(0, \frac{\sigma_\theta^2}{\gamma_P}\right)\right) - \mathbb{E}\left((V_\pi(z) - V_\pi(0))^2 \mid z \sim \mathcal{N}\left(0, \frac{\sigma_\theta^2}{\gamma_V}\right)\right) \\ &= \mathbb{E}\left((P_\pi(z) - P_\pi(0))^2 \mid z \sim \mathcal{N}\left(0, \frac{\sigma_\theta^2}{\gamma_V}\right)\right) - \mathbb{E}\left((P_\pi(z) - P_\pi(0))^2 \mid z \sim \mathcal{N}\left(0, \frac{\sigma_\theta^2}{\gamma_P}\right)\right) \\ &\quad + \text{Var}\left(P_\pi(z) \mid z \sim \mathcal{N}\left(0, \frac{\sigma_\theta^2}{\gamma_P}\right)\right) - \text{Var}\left(V_\pi(z) \mid z \sim \mathcal{N}\left(0, \frac{\sigma_\theta^2}{\gamma_V}\right)\right) \\ &\quad + (\mathbb{E}(\pi(\theta)) - P(0))^2 - (\mathbb{E}(\pi(\theta)) - V(0))^2. \end{aligned}$$

where we have just made explicit the distribution of z on which the expectation is taken, and we have added and subtracted the term $\mathbb{E}\left((P_\pi(z) - P_\pi(0))^2 \mid z \sim \mathcal{N}\left(0, \frac{\sigma_\theta^2}{\gamma_P}\right)\right)$ in the first equality, and then broken up the second line. In this expression, we have an amplification term, which captures the increased variance on z in the comparison of $\mathbb{E}\left((P_\pi(z) - P_\pi(0))^2\right)$ under $z \sim \mathcal{N}\left(0, \frac{\sigma_\theta^2}{\gamma_V}\right)$ and $z \sim \mathcal{N}\left(0, \frac{\sigma_\theta^2}{\gamma_P}\right)$ (first line), a variance term, which compares the variances of $P_\pi(z)$ and $V_\pi(z)$, and a bias term, which compares the biases between $P_\pi(0)$ and $V_\pi(0)$, relative to the unconditional average of $\pi(\cdot)$. We will analyze these three terms separately.

For the amplification term, we have

$$\begin{aligned} &\mathbb{E}\left((P_\pi(z) - P_\pi(0))^2 \mid z \sim \mathcal{N}\left(0, \frac{\sigma_\theta^2}{\gamma_V}\right)\right) - \mathbb{E}\left((P_\pi(z) - P_\pi(0))^2 \mid z \sim \mathcal{N}\left(0, \frac{\sigma_\theta^2}{\gamma_P}\right)\right) \\ &= \int (P_\pi(z) - P_\pi(0))^2 \left[\frac{\sqrt{\gamma_V}}{\sigma_\theta} \phi\left(\frac{\sqrt{\gamma_V}z}{\sigma_\theta}\right) - \frac{\sqrt{\gamma_P}}{\sigma_\theta} \phi\left(\frac{\sqrt{\gamma_P}z}{\sigma_\theta}\right) \right] dz \\ &= \int 2P'_\pi(z) (P_\pi(z) - P_\pi(0)) \left[\Phi\left(\frac{\sqrt{\gamma_P}z}{\sigma_\theta}\right) - \Phi\left(\frac{\sqrt{\gamma_V}z}{\sigma_\theta}\right) \right] dz \end{aligned}$$

For $z > 0$, $P_\pi(z) > P_\pi(0)$ and $\Phi(\sqrt{\gamma_P}z/\sigma_\theta) > \Phi(\sqrt{\gamma_V}z/\sigma_\theta)$, while for $z < 0$, both the inequalities are both reversed. It follows immediately that this integral is always positive.

For the variance term, we have, by standard arguments (again, Blackwell,...),

$$\text{Var}(\pi(\theta) | z \sim \mathcal{N}(0, \sigma_\theta^2)) > \text{Var}\left(P_\pi(z) | z \sim \mathcal{N}\left(0, \frac{\sigma_\theta^2}{\gamma_P}\right)\right) > \text{Var}\left(V_\pi(z) | z \sim \mathcal{N}\left(0, \frac{\sigma_\theta^2}{\gamma_V}\right)\right)$$

We are left with the bias term. Here, we notice that

$$\begin{aligned} \int_{-\infty}^{+\infty} \pi(\sqrt{1-\gamma}\sigma_\theta u) \phi(u) du &= \int_0^{+\infty} \left(\pi(\sqrt{1-\gamma}\sigma_\theta u) + \pi(-\sqrt{1-\gamma}\sigma_\theta u)\right) \phi(u) du \\ &= \pi(0) + \int_0^{+\infty} \left(\pi(\sqrt{1-\gamma}\sigma_\theta u) + \pi(-\sqrt{1-\gamma}\sigma_\theta u)\right) \phi(u) du \\ &= \pi(0) + \int_0^{+\infty} (\pi'(\theta) - \pi'(-\theta)) \left(1 - \Phi\left(\frac{\theta}{\sqrt{1-\gamma}\sigma_\theta}\right)\right) d\theta \end{aligned}$$

Applying this formula to $P(0)$ with $\gamma = \gamma_P$, to $V(0)$ with $\gamma = \gamma_V$, to $\mathbb{E}(\pi(\theta))$ with $\gamma = 0$, we find that for upside risks, $\mathbb{E}(\pi(\theta)) > V(0) > P(0) > \pi(0)$, while for downside risks, $\mathbb{E}(\pi(\theta)) < V(0) < P(0) < \pi(0)$. In both cases, $(\mathbb{E}(\pi(\theta)) - \pi(0))^2 > (\mathbb{E}(\pi(\theta)) - P(0))^2 > (\mathbb{E}(\pi(\theta)) - V(0))^2$, which completes the proof for part (i).

Part (ii) follows immediately from observing that these comparative statics apply separately to each of the terms used in the decomposition in part (i).

Part (iii) Fixing $\gamma_V < 1$. If $\gamma_P \rightarrow 1$, The variance and bias terms for P approach those for $\pi(\cdot)$. We thus wish to show merely that the amplification term doesn't vanish. But this term converges to

$$\int 2P'_\pi(z) (P_\pi(z) - P_\pi(0)) \left[\Phi\left(\frac{z}{\sigma_\theta}\right) - \Phi\left(\frac{\sqrt{\gamma_V}z}{\sigma_\theta}\right) \right] dz > 0.$$

Likewise, fixing $\gamma_P < 1$, as $\gamma_V \rightarrow 0$, the bias and variance terms are fixed, and we therefore consider the amplification term, which converges to

$$\begin{aligned} &\int 2P'_\pi(z) (P_\pi(z) - P_\pi(0)) \left[\Phi\left(\frac{\sqrt{\gamma_P}z}{\sigma_\theta}\right) - \frac{1}{2} \right] dz \\ &= \lim_{z \rightarrow \infty} \frac{1}{2} (P_\pi(z) - P_\pi(0))^2 + \lim_{z \rightarrow -\infty} \frac{1}{2} (P_\pi(-z) - P_\pi(0))^2 \\ &\quad - \mathbb{E}\left((P_\pi(z) - P_\pi(0))^2 | z \sim \mathcal{N}\left(0, \frac{\sigma_\theta^2}{\gamma_P}\right)\right), \end{aligned}$$

which is strictly positive, and infinite whenever $P_\pi(\cdot)$ (or equivalently $\pi(\cdot)$) is unbounded on at least one side (part iv). ■

Proof of Proposition 2. If $\sigma_{P,1} = \sigma_{P,2} = \sigma_P$, then $W_{\pi_1}(\sigma_{P,1}) + W_{\pi_2}(\sigma_{P,2}) = W_\pi(\sigma_P) = 0$, and

hence the total expected revenue is not affected by the split. If instead $\sigma_{P,1} \neq \sigma_{P,2}$, then

$$\begin{aligned} W_{\pi_1}(\sigma_{P,1}) + W_{\pi_2}(\sigma_{P,2}) &= W_{\pi_1}(\sigma_{P,1}) + W_{\pi_2}(\sigma_{P,1}) + W_{\pi_2}(\sigma_{P,2}) - W_{\pi_2}(\sigma_{P,1}) \\ &= W_{\pi}(\sigma_{P,1}) + W_{\pi_2}(\sigma_{P,2}) - W_{\pi_2}(\sigma_{P,1}). \end{aligned}$$

By theorem 1, $W_{\pi_2}(\cdot)$ is increasing in σ_P , so this term is positive, whenever $\sigma_{P,2} > \sigma_{P,1}$. ■

Proof of Proposition 3. For any alternative split (π_1, π_2) , the monotonicity requirements imply that $0 \leq \pi_1'(\theta) = \pi'(\theta) - \pi_2'(\theta) \leq \pi'(\theta)$. This in turn implies that for all $\theta \geq 0$, $\pi_1^{*'}(\theta) - \pi_1^{*'}(-\theta) = -\pi_1'(-\theta) \leq \pi_1'(\theta) - \pi_1'(-\theta)$ and $\pi_2^{*'}(\theta) - \pi_2^{*'}(-\theta) = \pi'(\theta) \geq \pi_2'(\theta) - \pi_2'(-\theta)$, i.e. π_1 has less downside risk and more upside risk than π_1^* , and π_2 has more downside risk and less upside risk than π_2^* . Moreover,

$$(\pi_1'(\theta) - \pi_1'(-\theta)) + (\pi_2'(\theta) - \pi_2'(-\theta)) = \pi'(\theta) - \pi'(-\theta) = (\pi_1^{*'}(\theta) - \pi_1^{*'}(-\theta)) + (\pi_2^{*'}(\theta) - \pi_2^{*'}(-\theta))$$

But then, the expected revenue of selling π_1 to the investor pool with $\sigma_{P,1}$ and π_2 to the investor pool with $\sigma_{P,2}$ is

$$W_{\pi_1}(\sigma_{P,1}) + W_{\pi_2}(\sigma_{P,2}) = W_{\pi}(\sigma_{P,1}) + W_{\pi_2}(\sigma_{P,2}) - W_{\pi_2}(\sigma_{P,1}),$$

while the expected revenue from selling π_1^* to the investor pool with $\sigma_{P,1}$ and π_2^* to the investor pool with $\sigma_{P,2}$ is

$$W_{\pi_1^*}(\sigma_{P,1}) + W_{\pi_2^*}(\sigma_{P,2}) = W_{\pi}(\sigma_{P,1}) + W_{\pi_2^*}(\sigma_{P,2}) - W_{\pi_2^*}(\sigma_{P,1}).$$

The difference in revenues is therefore $W_{\pi_2^*}(\sigma_{P,2}) - W_{\pi_2^*}(\sigma_{P,1}) - (W_{\pi_2}(\sigma_{P,2}) - W_{\pi_2}(\sigma_{P,1}))$, which is positive due to part (iv) of Theorem 1, and the fact that π_2^* contains more upside and less downside risk than π_2 , and $\sigma_{P,2} \geq \sigma_{P,1}$. ■

Proof of Proposition 5. If $\pi(\cdot)$ is convex, then by Theorem 1, for any finite \underline{w} , there exists $\hat{\delta} < 1$, s.t. $\delta > \hat{\delta}$, $\delta \mathbb{E}(w(z)) > -(1 - \delta)\underline{w}$. We therefore need to establish a lower bound for $w(z)$. But if $\pi(\cdot)$ is bounded below, then $\lim_{z \rightarrow -\infty} w(z) = 0$, and $w(z)$ is positive for sufficiently high z , so it is necessarily bounded. ■

Proof of Proposition 6. Part (i) is immediate. For part (ii), Rewrite $Var(P(y, z))$ as

$$Var(P(y, z)) = \sigma_{\theta}^2 \left(1 - \frac{\sigma_{\theta}^{-2}}{\sigma_{\theta}^{-2} + \alpha + \beta + \beta\sigma_u^{-2}} + \frac{\sigma_{\theta}^{-2}\beta(1 + \sigma_u^{-2})/\sigma_u^{-2}}{(\sigma_{\theta}^{-2} + \alpha + \beta + \beta\sigma_u^{-2})^2} \right)$$

and take FOCs w.r.t. α shows that $Var(P(y, z))$ is increasing in α if and only if $\frac{\sigma_u^{-2}}{2} > \frac{\beta + \beta\sigma_u^{-2}}{\sigma_\theta^{-2} + \alpha + \beta + \beta\sigma_u^{-2}}$.

For part (iii), rearrange $Var(W(y, z))$ as follows

$$\begin{aligned} Var(W(y, z)) &= \left(\frac{\sigma_\theta^{-2} + \alpha}{\sigma_\theta^{-2} + \alpha + \beta\sigma_u^{-2}} - \frac{\sigma_\theta^{-2} + \alpha}{\sigma_\theta^{-2} + \alpha + \beta + \beta\sigma_u^{-2}} \right)^2 \frac{\sigma_\theta^{-2} + \alpha + \beta\sigma_u^{-2}}{\beta\sigma_u^{-2}(\sigma_\theta^{-2} + \alpha)} \\ &= \left(\frac{\beta}{\sigma_\theta^{-2} + \alpha + \beta + \beta\sigma_u^{-2}} \right)^2 \frac{\sigma_\theta^{-2} + \alpha}{\sigma_\theta^{-2} + \alpha + \beta\sigma_u^{-2}} \frac{1}{\beta\sigma_u^{-2}} \end{aligned}$$

Taking the FOC, we find that $Var(W(y, z))$ is increasing in $\sigma_\theta^{-2} + \alpha$ if and only if

$$1 > 2 \frac{\sigma_\theta^{-2} + \alpha}{\sigma_\theta^{-2} + \alpha + \beta + \beta\sigma_u^{-2}} + \frac{\sigma_\theta^{-2} + \alpha}{\sigma_\theta^{-2} + \alpha + \beta\sigma_u^{-2}}.$$

■

Proof of Proposition 7. Risk-neutrality, position limits, and the MLRP on private signals jointly imply that $\mathbb{E}(\pi(\theta)|x, P)$ is monotone in x , and the informed demand is characterized by a private signal threshold $\hat{x}(P)$ which satisfies the same indifference condition as (2), with the posterior $H(\cdot|x, P)$ to be determined. Market-clearing then implies that $F(\hat{x}(P)|\theta) = D$. Since the LHS is increasing in $\hat{x}(P)$ and decreasing in θ , we can define an endogenous public signal $z = \hat{x}(P)$ which is informationally equivalent to observing P . But then, this implies that equilibrium prices take the form

$$P(z) = \mathbb{E}(\pi(\theta)|x = z, z) = \frac{\int \pi(\theta) h(\theta) \varphi(z|\theta) f(z|\theta) d\theta}{\int h(\theta) \varphi(z|\theta) f(z|\theta) d\theta},$$

where $\varphi(z|\theta)$ denotes the posterior density of z conditional on θ . To complete the proof notice that $\Pr(z \leq z'|\theta) = \Pr(D \leq F(z'|\theta)) = G(F(z'|\theta))$, from which it follows immediately that $\varphi(z|\theta) = g(F(z|\theta)) f(z|\theta)$. ■

Proof of Proposition 8.

If $P(z; 0) = V(z)$, then $P(z; \eta) = V(z)$ solves the pricing equation for any $\eta > 0$. If $P(z; 0) \neq V(z)$, then define the function $\mathcal{T}^\eta(P, z)$ as

$$\mathcal{T}^\eta(P, z) = \mathbb{E}\left(\pi(\theta) | z + \eta/\sqrt{\beta} \cdot (V(z) - P), z\right).$$

$\mathcal{T}^\eta(P, z)$ is continuous and decreasing in P , and $\mathcal{T}^\eta(V(z), z) = P(z; 0)$. Moreover, if $V(z) > P(z; 0)$, then $\mathcal{T}^\eta(P(z; 0), z) > P(z; 0)$, $\mathcal{T}^\eta(V(z), z) < V(z)$, and therefore there exists a unique $P(z; \eta) \in (P(z; 0), V(z))$, such that $\mathcal{T}^\eta(P(z; \eta), z) = P(z; \eta)$. If instead $V(z) < P(z; 0)$, then $\mathcal{T}^\eta(P(z; 0), z) < P(z; 0)$, $\mathcal{T}^\eta(V(z), z) > V(z)$, and $\mathcal{T}^\eta(P(z; \eta), z) = P(z; \eta)$ for a unique

$P(z; \eta) \in (V(z), P(z; 0))$. Moreover, replacing $P(z; 0)$ with $P(z; \eta')$ for $\eta' < \eta$ in the steps above shows that $|P(z; \eta) - V(z)|$ is strictly decreasing in η . For the limit, notice that since $P(z; \eta)$ is monotone in η and bounded, it must converge to a limit $P(z; \infty) = \lim_{\eta \rightarrow \infty} P(z; \eta)$. If $P(z; \infty) > V(z)$, then $P(z; \infty) = \lim_{\eta \rightarrow \infty} \mathcal{T}^\eta(P(z; \eta), z) = -\infty$, whereas if $P(z; \infty) < V(z)$, then $P(z; \infty) = \lim_{\eta \rightarrow \infty} \mathcal{T}^\eta(P(z; \eta), z) = \infty$, both of which are contradictions. Therefore, we are left with $P(z; \infty) = V(z)$ at the limit. ■

8 Appendix B: Additional signals

Here we solve an extension of our model in which traders in a market have access to an additional public signal from outside the market. Such a signal can either be a price in a separate market (as in the 2-market version of section 4.1, with informational linkages - lemma 3), or an exogenous public signal, as in the section with disclosures. We'll begin with a general structure and then specialize the results for the two separate cases. Suppose that two signals (z, y) and θ are jointly normally distributed according to

$$\begin{pmatrix} \theta \\ z \\ y \end{pmatrix} = \mathcal{N} \left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_\theta^2 & & \\ \sigma_\theta^2 & \sigma_\theta^2 + \tau_z^{-1} & \\ \sigma_\theta^2 & \sigma_\theta^2 + \rho\sqrt{\tau_z^{-1}\tau_y^{-1}} & \sigma_\theta^2 + \tau_y^{-1} \end{pmatrix} \right),$$

where τ_z^{-1} and τ_y^{-1} denote the precision levels of the two signals, and ρ the correlation in their errors (this corresponds to the additive signal structure used throughout). We will interpret signal z as coming from the price in the market, so that $\tau_z^{-1} = \sigma_u^2/\beta$. By Bayes' Rule, we have $\theta|z, y \sim \mathcal{N}(\mu(z, y), V^{-1})$, where

$$\mu(z, y) = \sigma_\theta^2 \mathbf{1}' \Sigma^{-1} \begin{pmatrix} z \\ y \end{pmatrix} =$$

$$\text{and } V^{-1} = \sigma_\theta^2 - \sigma_\theta^2 \mathbf{1}' \Sigma^{-1} \mathbf{1} \sigma_\theta^2 = \left\{ \sigma_\theta^{-2} + \frac{1}{1 - \rho^2} (\tau_z + \tau_y - 2\rho\sqrt{\tau_z\tau_y}) \right\}^{-1},$$

and $\mathbf{1}' = (1, 1)$. If $x \sim \mathcal{N}(\theta, \beta^{-1})$ is the traders' private signal distribution, then $\theta|x, z, y \sim \mathcal{N}(\hat{\mu}(x, z, y), (\beta + V)^{-1})$, where $\hat{\mu}(x, z, y) = (\beta x + V\mu(z, y)) / (\beta + V)$. Therefore,

$$\hat{\mu}(x = z, z, y) = (\beta + V)^{-1} (\beta' + V\sigma_\theta^2 \mathbf{1}' \Sigma^{-1}) \begin{pmatrix} z \\ y \end{pmatrix},$$

$$\text{where } \beta' = (\beta, 0) \text{ and } \Sigma = \begin{pmatrix} \sigma_\theta^2 + \tau_z^{-1} & \sigma_\theta^2 + \rho\sqrt{\tau_z^{-1}\tau_y^{-1}} \\ \sigma_\theta^2 + \rho\sqrt{\tau_z^{-1}\tau_y^{-1}} & \sigma_\theta^2 + \tau_y^{-1} \end{pmatrix}$$

From an ex ante perspective, $\hat{\mu}(x = z, z, y) \sim \mathcal{N}(0, \hat{\sigma}^2)$, where

$$\begin{aligned}
\hat{\sigma}^2 &= (\beta + V)^{-1} (\beta' + V\sigma_\theta^2 \mathbf{1}' \Sigma^{-1}) \Sigma (\Sigma^{-1} \mathbf{1} \sigma_\theta^2 V + \beta) (\beta + V)^{-1} \\
&= (\beta + V)^{-2} (\beta^2 (\sigma_\theta^2 + \tau_z^{-1}) + 2\beta\sigma_\theta^2 V + V\sigma_\theta^2 \mathbf{1}' \Sigma^{-1} \mathbf{1} \sigma_\theta^2 V) \\
&= (\beta + V)^{-2} (\beta^2 (\sigma_\theta^2 + \tau_z^{-1}) + 2\beta\sigma_\theta^2 V + V^2 \sigma_\theta^2 - V) \\
&= \sigma_\theta^2 + \left(\frac{\beta}{\beta + V} \right)^2 \tau_z^{-1} - \frac{V}{(\beta + V)^2},
\end{aligned}$$

Therefore, we compute σ_P^2 as $\sigma_P^2 = \hat{\sigma}^2 + (\beta + V)^{-1}$ or

$$\sigma_P^2 = \sigma_\theta^2 + \left(\frac{\beta}{\beta + V} \right)^2 \tau_z^{-1} - \frac{V}{(\beta + V)^2} + (\beta + V)^{-1} = \sigma_\theta^2 \left(1 + \frac{\beta\sigma_\theta^{-2}}{(\beta + V)^2} \frac{\tau_z + \beta}{\tau_z} \right)$$

For the 2-asset model of section 4.1, this formula can now be applied directly with $\tau_z = \beta_i / \sigma_{u,i}^2$, $\tau_y = \beta_{-i} / \sigma_{u,-i}^2$, and $\beta = \beta_i$, to find

$$\sigma_{P,i}^2 = \sigma_\theta^2 \left(1 + \frac{\beta\sigma_\theta^{-2}}{(\beta + V)^2} \frac{1 + \sigma_u^{-2}}{\sigma_u^{-2}} \right) \text{ where } V = \sigma_\theta^{-2} + \frac{1}{1 - \rho^2} \left(\frac{\beta_1}{\sigma_{u,1}^2} + \frac{\beta_2}{\sigma_{u,2}^2} - 2\rho \frac{\sqrt{\beta_1\beta_2}}{\sigma_{u,1}\sigma_{u,2}} \right)$$

which completes the proof of lemma 3. **switched notation between $\beta_{1,2}$ and β**

For the model with public disclosures, we apply the formula with $\tau_z = \beta\sigma_u^{-2}$ and $V = \sigma_\theta^{-2} + \beta\sigma_u^{-2} + \alpha$. Increased transparency always brings the average price closer to the average expected dividend, reducing unconditional discrepancies in returns.